Levy walk with rest: Escape from bouded domain

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Introduction

The Levy walk is a model of random walk which takes into account:
– a finite velocity of a random walker (in our case v=const.)
– a lenght of jump typical for Levy flights
– coupling between spatial and temporal characteristics of the system.
The time flight is given by coupled form of density distribution:

$$\bar{\psi}(\xi,\tau) = \frac{1}{2}\delta(|\xi| - v\tau)\psi(\tau). \quad (1)$$

which gives a relation between a jump length and the time of flight.
We assume the following form of $\psi(\tau)$:

$$\psi(\tau) = \begin{cases} \alpha\epsilon^{\alpha}\tau^{-1-\alpha} & \text{for } \tau > \epsilon \\ 0 & \text{for } \tau \leq \epsilon, \end{cases} \quad (2)$$

where $\epsilon =$ const. power law tails $0 < \alpha < 1$

Levy walk with rests

A Levy walk with rest between consecutive jumps will be considered. The time of rest is governed by exponential density distribution. Moreover, it will depend on function $\nu(x)$ -rate of jumps which depends on current position.

$$w(\tau) = \nu(x)e^{-\nu(x)\tau}, \tag{3}$$

The mean waiting time is $1/\nu(x)$.
Such defined model of Levy walk gives a wide class of processes, inter alia is able to be obtained Levy walk without a rest as a limit near the constance rate which goes to infinity.

Two phases

It can be noticed that the Levy walk processes with rests possess two types of phases: flight and rest.
– two density distributions: $p_v(x,t)$ and $p_r(x,t)$
The total density, $p(x,t) = p_r(x,t) + p_v(x,t)$, is normalised to unity but the contribution of individual phases to the total probability may change with time: for $\alpha < 1$, $p_r(x,t)$ decays and the flying phase prevails at long time.
We assume that the motion is restricted to the interval $(-L, L)$ by introducing absorbing barriers at $\pm L$ which means boundary conditions,

$$p(\pm L, t) = 0. \tag{4}$$

Density distribution of the rest phase

– the master equation for the rest phase,

$$\frac{\partial}{\partial t} p_r(x,t) =$$

$$-\nu(x)p_r(x,t)+\int_0^t\int \nu(x')p_r(x',t-t_1)\frac{1}{2}\psi(t_1)\delta(|x-x'|-vt_1)dt_1\,dx'. \quad (5)$$

Density distribution of the flight phase

the density corresponding to particles remaining in flight in x and at t,

$$p_v(x,t) = \int\int_0^t \nu(x')\Psi(t')\delta(|x-x'|-vt')p_r(x',t-t')dx'dt'. \qquad (6)$$

where $\Psi(t) = \int_t^\infty \psi(t')dt'$ is a survival probability
applying a property that $\delta'(x)$ is an odd function to evaluate the time derivative from $p_v(x,t)$,

$$\frac{\partial p_v(x,t)}{\partial t} = -\int\int_0^t \nu(x')\Psi(t')\delta(|x-x'|-vt')\partial p_r(x',t-t')/\partial t' dx'dt'. \qquad (7)$$

1. A. Kaminska and T. Srokowski, Lévy walks in nonhomogeneous environments, Phys. Rev. E, 96 (2017) 032105.

2. A. Kaminska and T. Srokowski, Lévy walks with variable waiting time: a ballistic case, Phys. Rev. E , 97 (2018) 062120

Rate function

As a $\nu(x)$ a constant and monotonic function of position were assumed. The results are qualitatively the same for constant and non strongly decreasing rate functions.
The strongly decreasing rate functions give very long rests in comparison with the time of fligt to the barriers and we have not a case with finit velocity of moving between cosecutive jumps (feature typical for Levy flights).

Limit equations

-taking the Fourier and Laplace transforms of eq.(5) and (7) and keeping the lowest terms in the expansion in powers of k and s
- limit of $s \to 0$

$$sp_r(k,s) - P_0(k) = -c_1\nu[s^\alpha + Bv^2k^2s^{\alpha-2}]p_r(k,s), \tag{8}$$

where $B = \alpha(1-\alpha)/2$ and $P_0(x)$ stands for an initial condition

$$sp_v(k,s) = c_1\nu\left[s^\alpha - s^{\alpha-2}\frac{1}{2}(1-\alpha)(2-\alpha)v^2k^2\right]p_r(k,s). \tag{9}$$

Limit equations

$$\frac{\partial p_r(x,t)}{\partial t} = -c_1\nu\left[\frac{\partial^2}{\partial t^2} - \frac{v^2}{B}(2-\alpha)\frac{\partial^2}{\partial x^2}\right] \times {}_0D_t^{\alpha-2}p_r(x,t). \qquad (10)$$

– the fractional differential equation

$$\frac{\partial p_v(x,t)}{\partial t} = -c_1\nu\left[\frac{\partial^2}{\partial t^2} + \frac{v^2}{2}(1-\alpha)(2-\alpha)\frac{\partial^2}{\partial x^2}\right] \times {}_0D_t^{\alpha-2}p_r(x,t). \qquad (11)$$

– partial differential equation, not fractional. We will solve this second equation with given initial and boundary conditions

Total density

the time evolution of the total density $p(x,t)$ can be determined from the expression,

$$\frac{\partial p(x,t)}{\partial t} = c_1 v^2 \nu (1-\alpha) \frac{\partial^2}{\partial x^2} {}_0D_t^{\alpha-2} p_r(x,t), \tag{12}$$

- results from the combining (8) with (9) the time evolution of the total density $p(x,t)$

Method of solving

It will be solved by a variable separation and evaluating eigenfunctions corresponding to both variables.

$$p_r(x,t) = \sum_{n=0}^{\infty} X_n(x) T_n(t), \tag{13}$$

$$p_v(x,t) = \sum_{n=0}^{\infty} X_n^{(v)}(x) T_n^{(v)}(t), \tag{14}$$

the position eigenfunctions of the term of nonhomogeneity the same form as the homogeneity part two possibilities: $X_n^{(v)}(x) = X_n(x)$ or $X_n^{(v)}(x) = -X_n(x)$

Equations for eigenfunctions

The separation of variables produces two equations that determine the eigenfunctions:

$$\frac{d^2}{dx^2}X_n(x) + \lambda_n X_n(x) = 0 \tag{15}$$

and

$$\frac{d^2}{dt^2}{}_0D_t^{\alpha-2}T_n(t) - C^2\lambda_{n0}D_t^{\alpha-2}T_n(t) = 0, \tag{16}$$

where $C = v\sqrt{(1-\alpha)(2-\alpha)}/2$

coresponding to the $X_n^{(v)}(x) = -X_n(x)$
and dependency has been included

$$\sum_n a_n \frac{dT_n^{(v)}(t)}{dt} = -\nu c_1 D_t^{\alpha} \sum_n a_n T_n(t) \tag{17}$$

eigenfunctions

The boundary conditions imply $X_n(-L) = X_n(L) = 0$ which yields the solution of Eq.(15) in the form,

$$X_n(x) = \cos(\sqrt{\lambda_n} x), \tag{18}$$

where the eigenvalues $\lambda_n = \pi^2 n^2 / 4L^2$ $(n = 1, 3, 5, \dots)$. Insertig the eigenvalues into Eq.(16) and solving the equation yields,

$${}_0D_t^{\alpha-2} T_n(t) = a_n e^{-C\sqrt{\lambda_n} t} + b_n e^{C\sqrt{\lambda_n} t}, \tag{19}$$

Solutions

and the solution of Eq.(11) is

$$\sum_n^{\infty} {}_0D_t^{\alpha-2}[X_n(x)T_n(t)] = \sum_n^{\infty} \Big[a_{2n+1}\exp(-\frac{C\pi(2n+1)t}{2L}) + b_{2n+1}\exp(\frac{C\pi(2n+1)t}{2L}) \Big] \cos\frac{\pi(2n+1)x}{2L}. \tag{20}$$

To evaluate the total density $p(x,t)$, the above result insert into Eq.(12) and the coefficients can be determined from the following conditions: $p(x,\infty)=0$ and the initial condition $p(x,0)=\delta(x)$ The final expression for the total density reads,

$$p(x,t) = \frac{1}{L}\sum_{n=0}^{\infty} \exp(-\frac{C\pi(2n+1)t}{2L}) \cos\frac{\pi(2n+1)x}{2L}. \tag{21}$$

Survival probability

The probability that the particle never reached those barriers up to time t: $S(t)=\int_{-L}^{L}p(x,t)dx$.

$$S(t)=\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\exp(-\frac{C\pi(2n+1)t}{2L}), \tag{22}$$

at large time is exponential. The differentiation of $S(t)$ and summation of the series yields the first passage time density $p_{FP}(t)=-dS(t)/dt$,

$$p_{FP}(t)=\frac{C}{L\cosh(C\pi t/2L)}, \tag{23}$$

MFPT

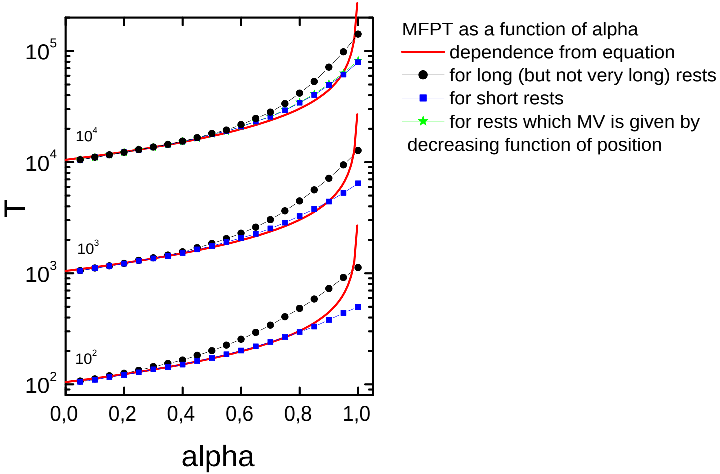
the integration of $S(t)$ yields MFPT: $T = \int_0^\infty t p_{FP}(t)dt = \int_0^\infty S(t)dt$

$$T = \frac{8L}{C\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} = \frac{16L}{v\pi^2}\frac{\mathcal{G}}{\sqrt{(1-\alpha)(2-\alpha)}}, \tag{24}$$

where $\mathcal{G} = 0.916\ldots$ is a Catalan constant

picture

MFPT as a function of α



Conclusions

We have discussed the Lévy walk with random waiting times between displacements and derived time characteristics of the escape process from a domain bounded by two absorbing barriers.
The combined density distribution for flights and rests, satisfying boundary conditions at barrier positions $\pm L$, has been evaluated by using solution of the Poisson equation; this equation determines a fractional operator from which the density evolution is derived.
The simple expression for MFPT has been obtained and dependences on L and α established (in particular, the proportionality of MFPT to L which dependence is well-known from numerical analyses of the problem without rests). The mean waiting time $1/\nu$, that enters the model as a parameter, establishes the relative duration of resting and moving. Therefore, the model incorporates both the case of Lévy walks process without rests (large ν) and the limit $\nu \to 0$ when the time of flight needed to reach the barrier becomes negligible compared to the resting time. This case reveals features typical for Lévy flights, in particular, the dependence $MFPT \propto L^\alpha$.