Warsaw University of Technology
Faculty of Physics

Koszykowa 75, PL-00-662 Warsaw, Poland
Tel: (48) 22 2347267; fax: (48) 22 6282171; http://www.fizyka.pw.edu.pl

Andrzej Krawiecki, Tomasz Gradowski

q-Neighbor majority vote model on complex networks

32 M. Smoluchowski Symposium on Statistical Physics
Kraków, Poland, September 18-20, 2019

Motivation

- The majority vote (MV) model on networks, in which the probability that an agent changes opinion depends only on the sign of the resultant opinion of all his/her neighbors, is a simple nonequilibrium model for the opinion formation.

- In general, the MV model exhibits transition from the disordered (paramagnetic, PM) phase to the ordered (ferromagnetic, FM) phase as the degree of agents' randomness in decision making is decreased. This transition is usually second-order.

- There are many other models for the opinion formation on networks (e.g., the q-voter model with independence, q-neighbor Ising model) in which the probability that an agent changes opinion is a function of the opinions of a subset of q his/her neighbors (q-panel). In such models both second- and first-order FM transitions were reported.

- It was argued that a necessary condition for the first-order transition in certain models for the opinion formation is that the opinion update rule includes a sort of threshold (only sufficiently large majority of neighbors with opposite opinion can increase the probability that a given agent changes his/her opinion).

- Thus it seems interesting to investigate a q-neighbor version of the MV model to find the possible FM transition and its order.

The majority vote (MV) model [M. J. Oliveira, *J. Stat. Phys*. **66**, 273 (1992)]

The MV model consists of two-state spins $\sigma_j = \pm 1$, $j = 1, 2, \ldots N$, located in the nodes of a network of interactions with the distribution of the degrees of nodes $P(k)$.

The probability per unit time (rate) that the spin σ_j in node j flips is

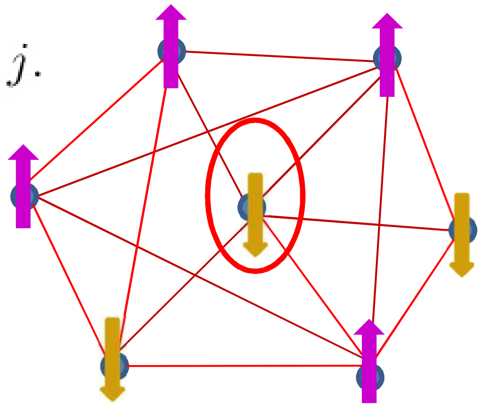
$$w_j(\sigma) = \frac{1}{2}\left[1 - (1-2p)\sigma_j \mathrm{sign}_j\right],$$

where σ denotes the spin configuration, p is the internal noise parameter,

$$\mathrm{sign}_j = \mathrm{sign}\left(\sum_{j' \in nn_j} \sigma_{j'}\right), \quad \mathrm{sign}(x) = \begin{cases} -1 & \text{for} \quad x < 0 \\ 0 & \text{for} \quad x = 0 \\ +1 & \text{for} \quad x > 0, \end{cases}$$

and nn_j denotes a set of the nearest neighbors of the node j.

The MV model is a nonequilibrium model equivalent to the Ising model with Glauber dynamics with random contact with two thermal baths, one with $T = 0$ (with probability $1 - 2p$) and the other with $T \to \infty$ (with probability $2p$), $0 \le p \le 1/2$.



The q-neighbor majority vote (q-MV) model

In the q-neighbor MV (q-MV) model at each elementary simulation step the node j and a set of its q neighbors (q-neighborhood) out of its k_j neighbors is chosen randomly.

The probability per unit time (rate) that the spin σ_j in node j flips is

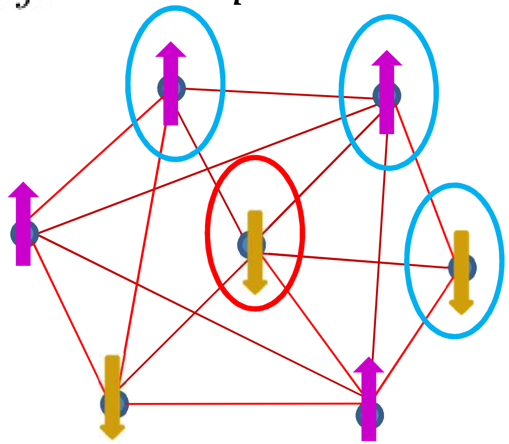
$$w_j(\sigma) = \frac{1}{2}\left[1 - (1-2p)\sigma_j \mathrm{sign}_j\right],$$

where

$$\mathrm{sign}_j = \mathrm{sign}\left(\sum_{j' \in nn_{j,q}} \sigma_{j'}\right),$$

and $nn_{j,q}$ denotes the selected q-neighborhood of the node j.

$q = 3$



Related models:

- *q*-voter model with independence [P.Nyczka, K. Sznajd-Weron and J. Cisło, *Phys. Rev. E* **86**, 011105 (2012); A. Jędrzejewski, *Phys. Rev. E* **95**, 012307 (2017)].
- *q*- neighbor Ising model [A. Jędrzejewski, A. Chmiel and K. Sznajd-Weron, *Phys. Rev. E* **92**, 052105 (2015); A. Chmiel, T. Gradowski and A. Krawiecki, *Int. J. Modern Phys. C* **29**, 1850041 (2018)].

Mean field approximation (MFA)

The probability $P(\sigma, t)$ that at time t the spin configuration is σ obeys the Master equation

$$\frac{dP(\sigma,t)}{dt} = \sum_{\sigma'} \left[w(\sigma|\,\sigma') P(\sigma',t) - w(\sigma'|\,\sigma) P(\sigma,t) \right].$$

At each time step transition occurs between spin configurations $\sigma' \to \sigma$ differing just by one spin flipped, thus the transition rate

$$w(\sigma|\,\sigma') = w_j(\sigma')$$

Performing ensemble average of the Master equation it is obtained that

$$\frac{\partial \langle \sigma_j \rangle}{\partial t} = -2\langle \sigma_j w_j(\sigma) \rangle = -\langle \sigma_j \rangle + (1-2p)\langle \mathrm{sign}_j \rangle,$$

where $\langle \sigma_i \rangle \equiv m$ is the mean value of the spin in node j (magnetization)

$$\langle \mathrm{sign}_j \rangle = (+1)\,\mathrm{Pr}\left(\mathrm{sign}_j = +1\right) + (-1)\,\mathrm{Pr}\left(\mathrm{sign}_j = -1\right)$$

$$\mathrm{Pr}\left(\mathrm{sign}_j = -1\right) = \mathrm{Pr}\left(\sum_{j' \in \mathrm{nn}_{q,j}} \sigma_{j'} < 0\right) =$$

$$= \frac{1}{\binom{k_i}{q}} \sum_{\binom{k_i}{q} \text{ choices of } \mathrm{nn}_{q,j}} \sum_{l < \frac{q}{2}} \binom{q}{l} \left[\mathrm{Pr}\left(\sigma_{j'} = +1\right)\right]^l \left[\mathrm{Pr}\left(\sigma_{j'} = -1\right)\right]^{q-l}$$

$$= \sum_{l < \frac{q}{2}} \binom{q}{l} \left(\frac{1+m}{2}\right)^l \left(\frac{1-m}{2}\right)^{q-l},$$

etc., thus

$$\frac{dm}{dt} = -m + (2p-1)\left[\sum_{l < \frac{q}{2}} \binom{q}{l} \left(\frac{1+m}{2}\right)^l \left(\frac{1-m}{2}\right)^{q-l} - \sum_{l > \frac{q}{2}} \binom{q}{l} \left(\frac{1+m}{2}\right)^l \left(\frac{1-m}{2}\right)^{q-l}\right]$$

The above equation in the MFA has a paramagnetic (PM) fixed point $m = 0$

For large q the binomial distribution can be approximated by the normal distribution

- for q odd $$\sum_{l=0}^{\frac{q-1}{2}} \binom{q}{l} \xi^l (1-\xi)^{q-l} \approx \frac{1}{2} - \frac{1}{2}\mathrm{erf}\left[\frac{\xi-\frac{1}{2}}{\sqrt{\xi(1-\xi)}}\sqrt{\frac{q}{2}}\right] \approx \frac{1}{2} - \frac{1}{2}\mathrm{erf}\left[\left(\xi-\frac{1}{2}\right)\sqrt{2q}\right]$$

(where the last approximation is valid for $\xi \approx 1/2$)

- for q even $$\sum_{l=0}^{\frac{q}{2}-1} \binom{q}{l} \xi^l (1-\xi)^{q-l} \approx \frac{1}{2} - \frac{1}{2}\mathrm{erf}\left[\left(\xi-\frac{1}{2}\right)\sqrt{2q}\right] - \frac{1}{2}\binom{q}{\frac{q}{2}} \xi^{\frac{q}{2}} (1-\xi)^{\frac{q}{2}}$$

The approximate equation for the magnetization in the MFA

$$\frac{dm}{dt} = -m + (2p-1)\mathrm{erf}\left(\sqrt{\frac{q}{2}}m\right)$$

The above equation has a PM fixed point $m = 0$ which becomes unstable for $p < p_c$, where

$$p_c = \frac{1}{2}\left(1 - \frac{\sqrt{2\pi}}{2}q^{-1/2}\right).$$

It is interesting to note that this is the same result as for the MV model on RRG with $k_i = const = q$.

Monte Carlo (MC) simulations

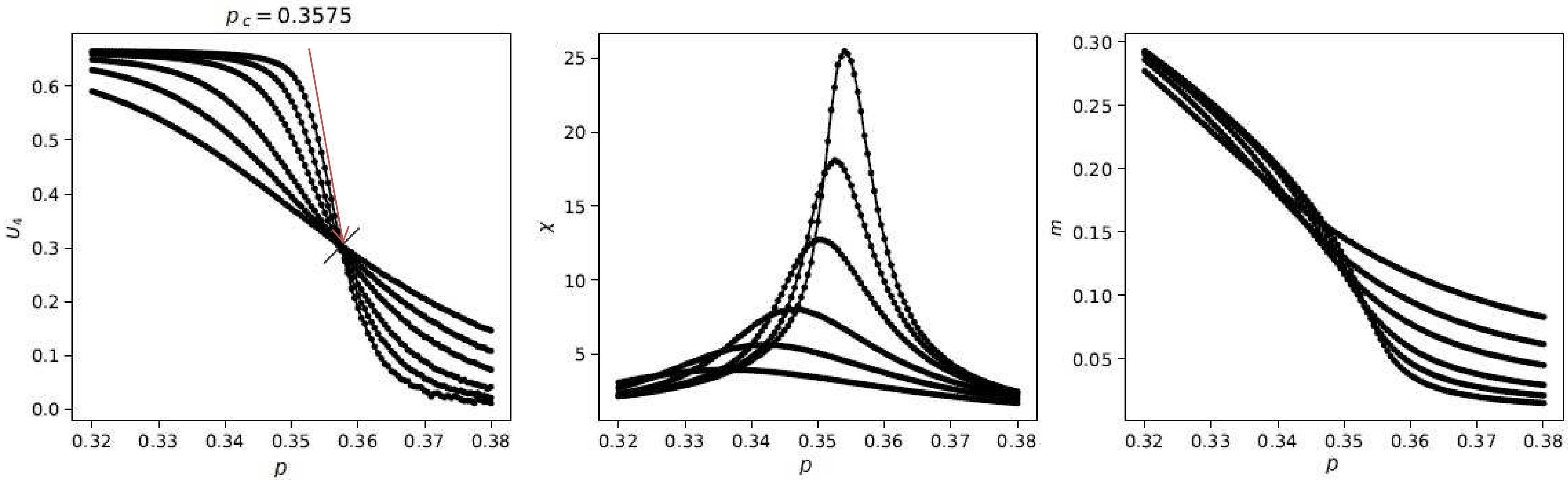


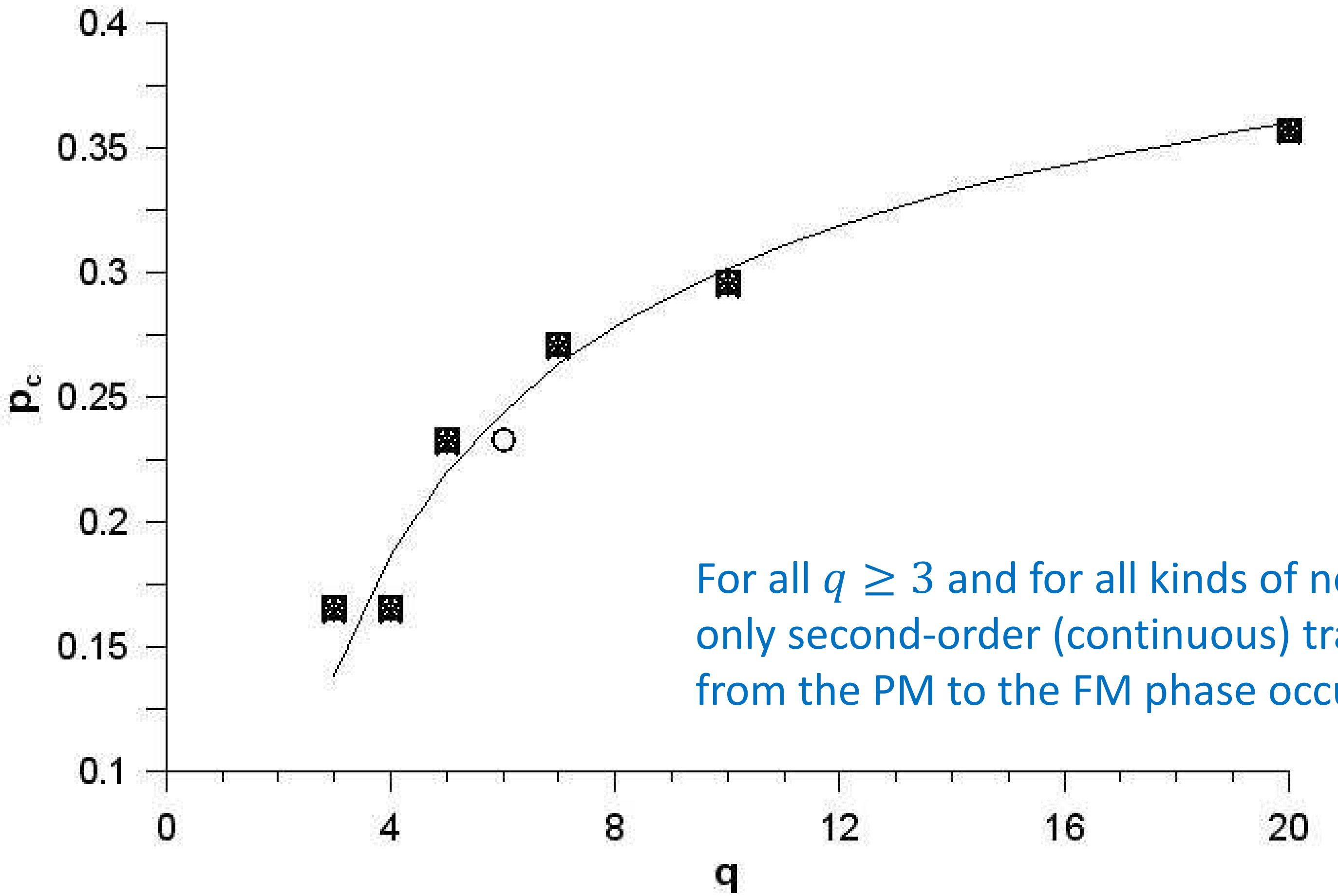
Figure 1: The Binder cumulant U_4, the susceptibility χ and the magnetization M vs. p for the q-MV model with $q = 20$ on SF networks with $\tilde{\gamma} = 5.0$, $\langle k \rangle = 100$ and $N = 5 \cdot 10^2$, 10^3, $2 \cdot 10^3$, $5 \cdot 10^3$, 10^4, $2 \cdot 10^4$ (in all cases curves with smaller maximum curvature correspond to smaller N).

Magnetization
$$M(q) = [\langle |\tilde{m}| \rangle_t]_{av}, \quad \text{where} \quad \tilde{m} = N^{-1} \sum_{i=1}^{N} \sigma_i, \tag{31}$$

Magnetic susceptibility
$$\chi(q) = N \left[\left(\langle \tilde{m}^2 \rangle_t - \langle |\tilde{m}| \rangle_t^2 \right) \right]_{av}, \tag{32}$$

Binder cumulant
$$U_4(q) = \frac{1}{2} \left[3 - \frac{\langle \tilde{m}^4 \rangle_t}{\langle \tilde{m}^2 \rangle_t^2} \right]_{av}, \tag{33}$$

Critical value of the noise parameter for the ferromagnetic transition



For all $q \geq 3$ and for all kinds of networks only second-order (continuous) transition from the PM to the FM phase occurs

Figure 2: The critical value p_c vs. q for the q-MV model on networks with mean degree $\langle k \rangle = 100$, symbols: results obtained from MC simulations for RRG (circles), ERG (squares), SF networks with $\tilde{\gamma} = 5$ (triangles), SF networks with $\tilde{\gamma} = 3.0$ (pluses), SF networks with $\tilde{\gamma} = 2.5$ (crosses), solid line: theoretical result obtained in the MF approximation.

Finite size scaling (FSS) analysis

$$
\begin{aligned}
M &= N^{-\beta/\nu} f_M\left(N^{1/\nu}(p - p_c)\right) \\
\chi &= N^{\gamma/\nu} f_\chi\left(N^{1/\nu}(p - p_c)\right) \\
p_c - p^\star(N) &\propto N^{-1/\nu},
\end{aligned}
$$

where $p^\star(N)$ denotes the value of p for which the susceptibility χ of the model on a network with N nodes has a maximum value.

The MFA suggests $\beta = 1/2$ for all kinds of networks (i.e., the same scaling exponent as for the Ising model in the MFA)

Hyperscaling relation

$$
2\frac{\beta}{\nu} + \frac{\gamma}{\nu} = D_{eff},
$$

where the effective dimension $D_{eff} = 1$ is expected in the case of systems on complex networks

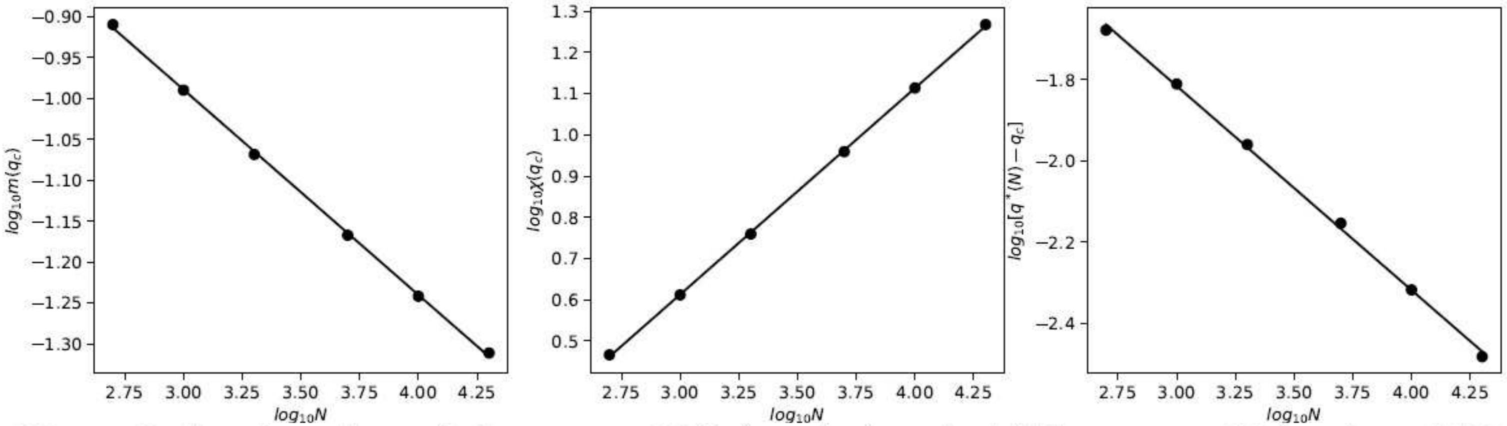


Figure 3: Log-log plots of the curves $M(p_c)$, $\chi(p_c)$ and $p^{\star}(N) - p_c$ vs. N for the q-MV model with $q = 20$ on SF networks with $\tilde{\gamma} = 5.0$, $\langle k \rangle = 100$ and $N = 5 \cdot 10^2$, 10^3, $2 \cdot 10^3$, $5 \cdot 10^3$, 10^4, $2 \cdot 10^4$.

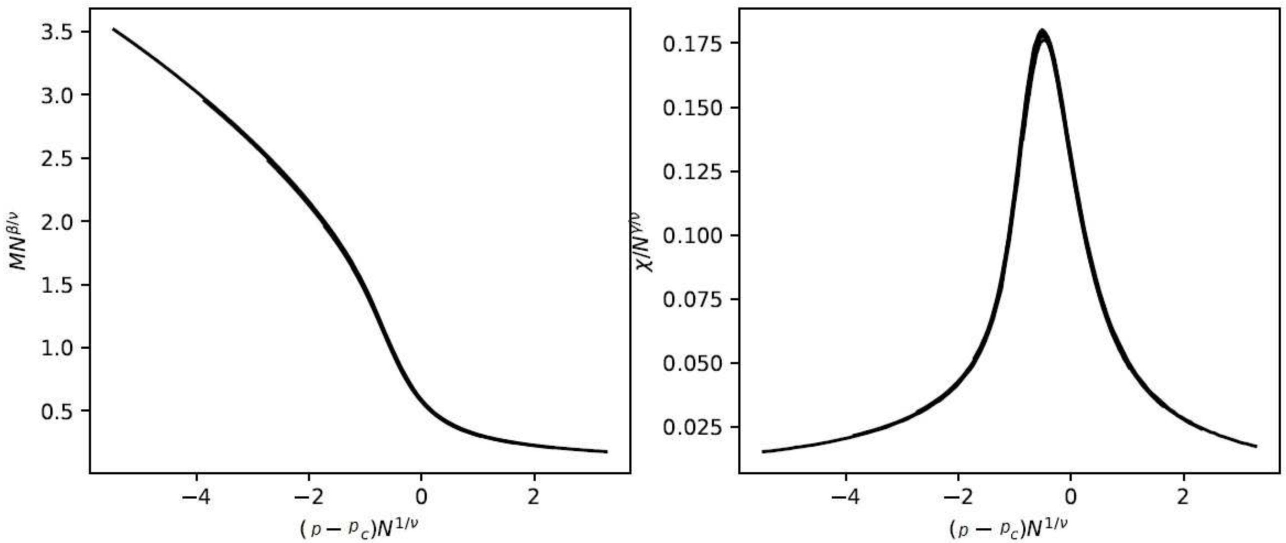


Figure 4: Rescaled magnetization $MN^{\beta/\nu}$ and susceptibility $\chi N^{-\gamma/\nu}$ vs. $(p - p_c) N^{1/\nu}$ for the q-MV model with $q = 20$ on SF networks with $\tilde{\gamma} = 5.0$,$\langle k \rangle = 100$ and $N = 5 \cdot 10^2$, 10^3, $2 \cdot 10^3$, $5 \cdot 10^3$, 10^4, $2 \cdot 10^4$.

Table I. The critical value p_c, exponents β/ν, γ/ν, $1/\nu$, effective dimension D_{eff} and the critical exponent β for the q-MV model with $q = 20$ on different networks with $\langle k \rangle = 100$

m	q_c	β/ν	γ/ν	$1/\nu$	D_{eff}	β
RRG	0.3575	0.254(3)	0.491(3)	0.510(8)	0.999	0.498
ERG	0.3575	0.252(4)	0.494(4)	0.513(10)	0.998	0.490
SF $\tilde{\gamma} = 5$	0.3575	0.251(3)	0.500(3)	0.503(9)	1.002	0.499
SF $\tilde{\gamma} = 3$	0.357	0.197(2)	0.610(4)	0.404(11)	1.003	0.487
SF $\tilde{\gamma} = 2.5$	0.3565	0.144(6)	0.717(13)	0.274(5)	1.005	0.527

For SF networks with $\tilde{\gamma} < 3$ there is $\frac{1}{\nu} \approx \frac{1}{2}\left(1 - \frac{3-\tilde{\gamma}}{\tilde{\gamma}-1}\right) = \frac{1}{3}$ (prediction for the q-voter model with independence [A.F. Peralta et al., Chaos **28**, 075516 (2018)])

Critical value q_c does not depend on the degree distribution

For weakly heterogeneous networks the model belongs to the Ising MF universality class

In all cases $\beta \approx 1/2$ as predicted in the MFA

Pair approximation (PA)

$c = (M+1)/2$ concentration of spins with orientation up

$1-c$ concentration of spins with orientation down

b concentration of active edges (bonds) connecting spins with opposite orientations

$$\theta_\uparrow = \frac{b}{2c} \qquad \theta_\downarrow = \frac{b}{2(1-c)}$$

conditional probabilities that an active link is selected provided that a node with spin up (resp. down) was selected as first

For any system on a network with the degree distribution $P(k)$ characterized by the spin-flip rate $f_k(i,q,p)$ dependent on the number of active bonds i the rate equations for the macroscopic quantities c, b in the PA are

$$\frac{\partial c}{\partial t} = \gamma^+ - \gamma^-,$$

$$\frac{\partial b}{\partial t} = \frac{2}{\langle k \rangle} \sum_{\nu \in \{\uparrow,\downarrow\}} c_\nu \sum_k P(k) \sum_{i=0}^{k} \binom{k}{i} \theta_\nu^i (1-\theta_\nu)^{k-i} f_k(i,q,p)(k-2i).$$

[A. Jędrzejewski, *Phys. Rev. E* **95**, 012307 (2017)].

PA for the q-MV model on networks

For a spin in a node with degree k with i active bonds the flip rate $f_k(i,q,p)$ is

$$f_k(i,q,p) = \frac{1}{\binom{k}{q}} \sum_{l=0}^{q} \binom{i}{l} \binom{k-i}{q-l} w(l,q,p) = \frac{1}{\binom{k}{i}} \sum_{l=0}^{q} \binom{k-q}{i-l} \binom{q}{l} w(l,q,p),$$

where

$$w(l,q,p) = \begin{cases} 1-p & \text{for} \quad l > q/2 \\ 1/2 & \text{for} \quad l = q/2 \\ p & \text{for} \quad l < q/2. \end{cases}$$

The rate equations then become (cf. analogous equations for the q-neighbor Ising model [A. Chmiel, T. Gradowski and A. Krawiecki, *Int. J. Modern Phys. C* **29**, 1850041 (2018)])

$$\frac{\partial c}{\partial t} = \sum_{l=0}^{q} \binom{q}{l} \left[(1-c)\theta_{\downarrow}^{l} (1-\theta_{\downarrow})^{q-l} - c\theta_{\uparrow}^{l} (1-\theta_{\uparrow})^{q-l} \right] w(l,q,p),$$

$$\frac{\partial b}{\partial t} = \frac{2}{\langle k \rangle} \sum_{\nu \in \{\uparrow,\downarrow\}} c_{\nu} \sum_{l=0}^{q} \binom{q}{l} \theta_{\nu}^{l} (1-\theta_{\nu})^{q-l} \left[\langle k \rangle - 2\left(\langle k \rangle - q\right)\theta_{\nu} - 2l \right] w(l,q,p).$$

For large *q*, after approximating the binomial distribution by the normal distribution in the vicinity of the PM fixed point , and using the Stirling formula, the rate equations become

$$\begin{aligned} \frac{\partial c}{\partial t} &= (1-c)\left\{\frac{1}{2}+\frac{1}{2}(1-2p)\mathrm{erf}\left[\left(\theta_{\downarrow}-\frac{1}{2}\right)\sqrt{2q}\right]\right\} \\ &- c\left\{\frac{1}{2}+\frac{1}{2}(1-2p)\mathrm{erf}\left[\left(\theta_{\uparrow}-\frac{1}{2}\right)\sqrt{2q}\right]\right\}, \end{aligned}$$

$$\begin{aligned} \frac{\partial b}{\partial t} &= \frac{2}{\langle k\rangle}\sum_{\nu\in\{\uparrow,\downarrow\}} c_{\nu}\left\{\left[\frac{1}{2}+\frac{1}{2}(1-2p)\mathrm{erf}\left(\left(\theta_{\nu}-\frac{1}{2}\right)\sqrt{2q}\right)\right]\langle k\rangle\left(1-2\theta_{\nu}\right)\right. \\ &- \left.(1-2p)\frac{2^{q}q}{\sqrt{2\pi q}}\theta_{\nu}^{\frac{q}{2}}\left(1-\theta_{\nu}\right)^{\frac{q}{2}}\right\}. \end{aligned}$$

The fixed point of the system of equations for c, b in the PA corresponding to the PM solution is

$$c = 1/2, \quad \theta_\uparrow = \theta_\downarrow \equiv \theta = b, \quad 0 < b < 1/2,$$

where θ is a non-zero solution of a nonlinear equation

$$\left\{ \frac{1}{2} + \frac{1}{2}(1-2p)\mathrm{erf}\left[\left(\theta - \frac{1}{2} \right) \sqrt{2q} \right] \right\} \langle k \rangle (1-2\theta) = (1-2p) \frac{2^q q}{\sqrt{2\pi q}} \theta^{\frac{q}{2}} (1-\theta)^{\frac{q}{2}} .$$

Performing linear stability analysis it can be shown that the PM fixed point loses stability at

$$p_c = \frac{1}{2}\left(1 - \frac{1}{2\Theta(\theta^\star)} \right),$$

where

$$\Theta(\theta) = \sqrt{\frac{2q}{\pi}} \theta \exp\left[-2q \left(\theta - \frac{1}{2} \right)^2 \right] - \frac{1}{2}\mathrm{erf}\left[\left(\theta - \frac{1}{2} \right) \sqrt{2q} \right],$$

and $\theta^\star$ is a non-zero solution of a nonlinear equation

$$\theta^\star \exp\left[-2q \left(\theta^\star - \frac{1}{2} \right)^2 \right] (1-2\theta^\star) = 2^{q-1} \langle k \rangle^{-1} \theta^{\star \frac{q}{2}} (1-\theta^\star)^{\frac{q}{2}} .$$

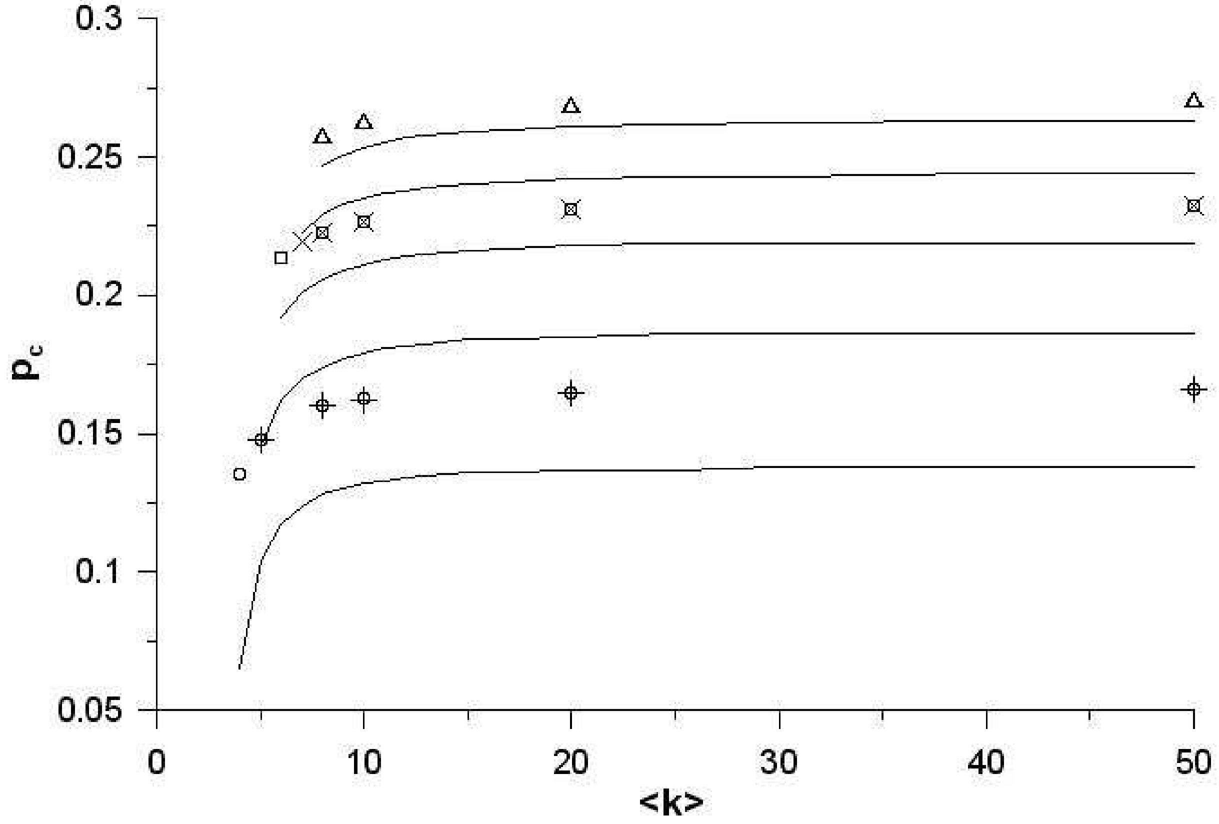


Figure 5: The critical value p_c vs. $\langle k \rangle$ for the q-MV model on RRGs with different q, symbols: results obtained from MC simulations for $q = 3$ (circles), $q = 4$ (pluses), $q = 5$ (squares), $q = 6$ (crosses), $q = 7$ (triangles), solid lines: theoretical results obtained in the PA for the above values of q (from bottom to top).

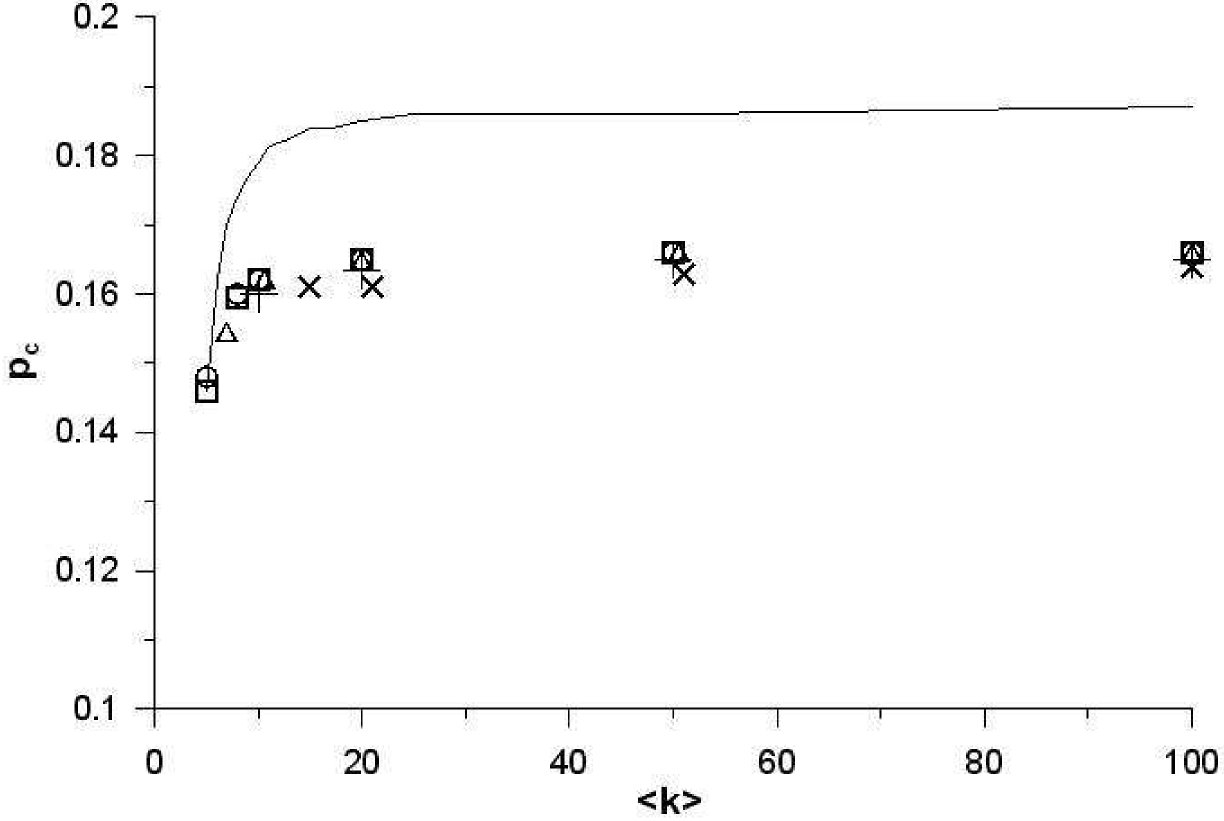


Figure 6: The critical value p_c vs. $\langle k\rangle$ for the q-MV model with$q = 4$ on different complex networks, symbols: results obtained from MC simulations for RRG (circles), ERG (squares), SF network with $\tilde{\gamma} = 5$ (triangles), SF network with $\tilde{\gamma} = 3.0$ (pluses), SF network with $\tilde{\gamma} = 2.5$ (crosses) solid line: theoretical result obtained in the PA.

Generalization: q-MV model with inertia (MFA)

For the q-neighbor MV model with inertia the spin flip rate is

$$w_j(\sigma) = \frac{1}{2}\left[1 - (1-2p)\sigma_j \mathrm{sign}_j\right],$$

with

$$\mathrm{sign}_j = \mathrm{sign}\left((1-\eta)\frac{1}{q}\sum_{j' \in \mathrm{nn}_{q,j}} \sigma_{j'} + \eta\sigma_j\right).$$

Equation for the magnetization $m \equiv \langle\sigma_j\rangle$ is

$$\frac{\partial\langle\sigma_j\rangle}{\partial t} = -2\langle\sigma_j w_j(\sigma)\rangle = -\langle\sigma_j\rangle + (1-2p)\langle\mathrm{sign}_j\rangle,$$

$$\langle\mathrm{sign}_j\rangle = (+1)\Pr\left(\mathrm{sign}_j = +1\right) + (-1)\Pr\left(\mathrm{sign}_j = -1\right).$$

$$\begin{aligned}\Pr\left(\mathrm{sign}_j = -1\right) &= \Pr\left(\mathrm{sign}_j = -1 \middle| \sigma_j = +1\right)\Pr\left(\sigma_j = +1\right) \\ &+ \Pr\left(\mathrm{sign}_j = -1 \middle| \sigma_j = -1\right)\Pr\left(\sigma_j = -1\right),\end{aligned}$$

q-neighbor version of the MV model with inertia [Hanshuang Chen et al., *Phys Rev E* **95**, 042304 (2017)] or an equivalent Watts model (with threshold) [B. Nowak and K. Sznajd-Weron, *Complexity* 5150825 (2019)]

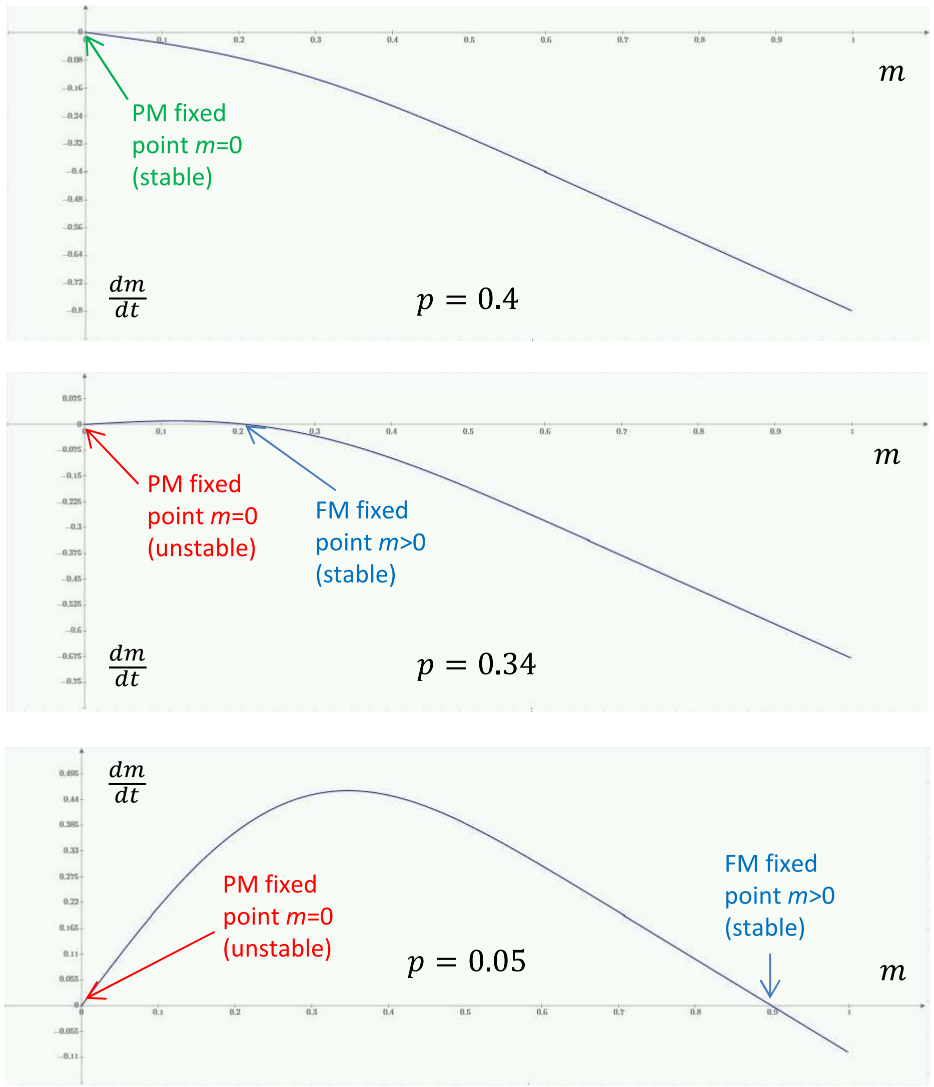
$$\Pr\left(\mathrm{sign}_j = -1 \middle| \sigma_j = -1\right) = \Pr\left(\sum_{j' \in \mathrm{nn}_{q,j}} \sigma_{j'} < \frac{\eta q}{1-\eta}\right) = \sum_{l < \frac{q}{2}\frac{1}{1-\eta}} \binom{q}{l} \left(\frac{1+m}{2}\right)^{l} \left(\frac{1-m}{2}\right)^{q-l},$$

etc. Approximating for large q the binomial distribution by the normal distribution the following equation for the magnetization m is obtained

$$\frac{dm}{dt} = -\left\{1 - \frac{1-2p}{2}\left[\mathrm{erf}\left(\sqrt{\frac{q}{2}}\frac{m+\frac{\eta}{1-\eta}}{\sqrt{1-m^2}}\right) - \mathrm{erf}\left(\sqrt{\frac{q}{2}}\frac{m-\frac{\eta}{1-\eta}}{\sqrt{1-m^2}}\right)\right]\right\} m + \frac{1-2p}{2}\left[\mathrm{erf}\left(\sqrt{\frac{q}{2}}\frac{m+\frac{\eta}{1-\eta}}{\sqrt{1-m^2}}\right) + \mathrm{erf}\left(\sqrt{\frac{q}{2}}\frac{m-\frac{\eta}{1-\eta}}{\sqrt{1-m^2}}\right)\right].$$

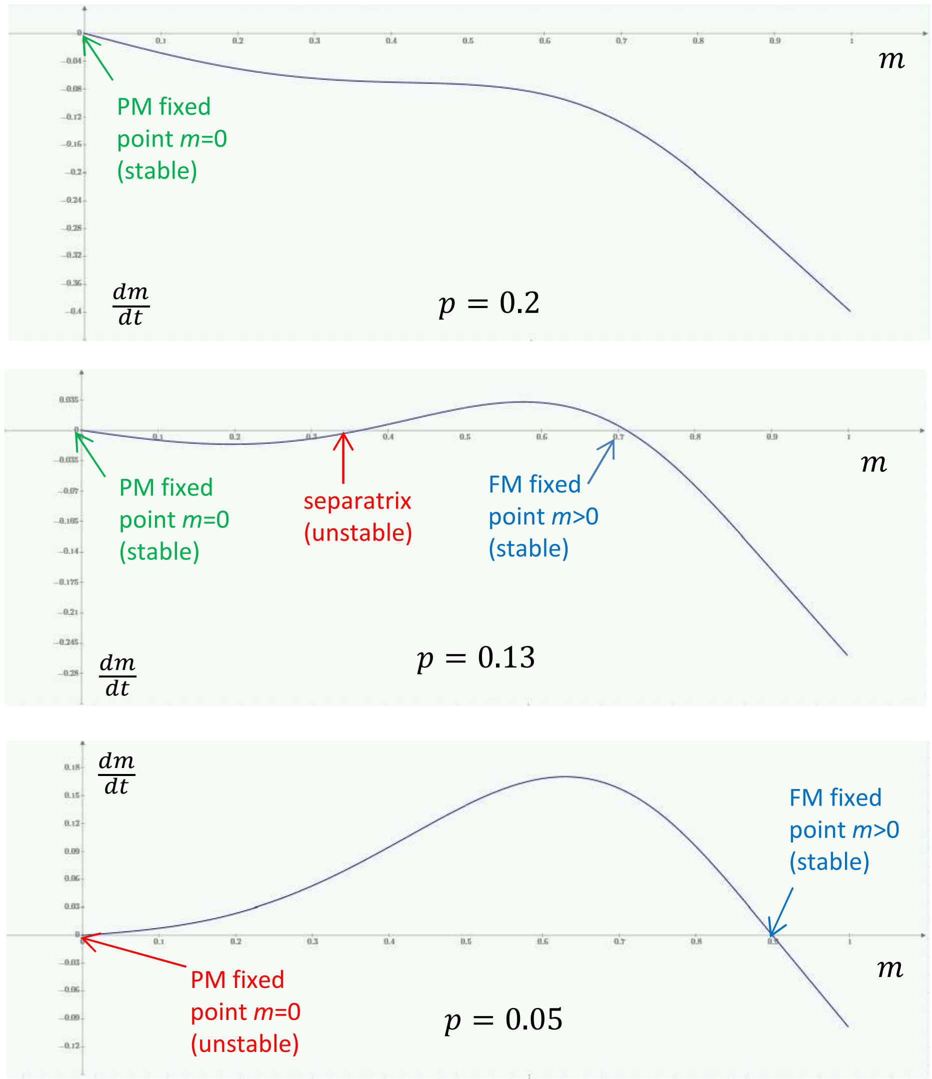
- For $\eta = 0$ this model is equivalent to the *q*-MV model,
- For $\eta \to 1/2$ this model is equivalent to the *q*-voter model with independence,
- For $\eta > \eta_c(q)$ this model can exhibit first-order transition from the PM to the FM phase with decreasing *p,*
- PA for the *q*-MV model can be easily generalizad to the *q*-MV model with inertia.

$$q = 20, \qquad \theta = 0.10$$



Second-order transition

$$q = 20, \qquad \theta = 0.35$$



First-order transition

Conclusions

- The q-neighbor majority vote model on networks was studied numerically, by means of Monte Carlo simulations, and theoretically, using the mean field and pair approximations.
- Only second-order FM transition was observed as the model parameter p was decreased.
- Both MFA and PA (within their limits of validity) predict correctly the critical value p_c of the model parameter.
- Finite size scaling analysis reveals that (for large q and $\langle k \rangle$) the q-MV model on homogeneous and weakly heterogeneous networks belongs to the Ising Mfuniversality class, while for the model on strongly heterogeneous networks the ecponent ν is non-universal and depends on the degree distribution.
- Introduction of inertia (threshold in the opinion flip rate) in the model allows for the occurrence of both first- and second-order FM transition. The q-MV and q-voter model with independence are special limiting cases of this more general model.


Thank you for your attention
www.if.pw.edu.pl/~akraw
andrzej.krawiecki@pw.edu.pl
2009/9/15 13:36