Stochastic Resetting

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Search Problems

Search problems are ubiquitous in nature

- search for Holy Grail
- search for Higgs boson

  . . .

  . . .

- data search (Google)
- animals searching for food (foraging)
- protein searching for a binding site on a DNA
- Visual search: locating a face in the crowd

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A robust class of models: Intermittent target search strategies combine

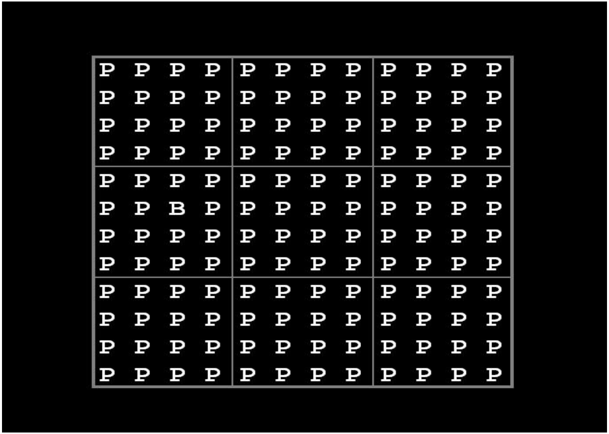
(i) phases of slow motion (target detection)

(ii) phases of fast motion (searcher relocates but not reactive)

[O. Bénichou et. al. Rev. Mod. Phys. 83, 81 (2011)]

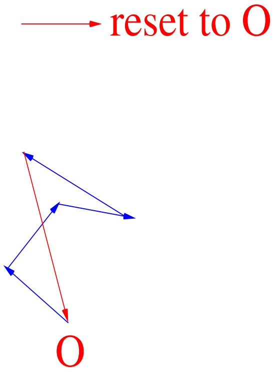
Visual search: a face in a crowd

Visual search in psychology



Search via diffusion and resetting

Schematic search trajectory



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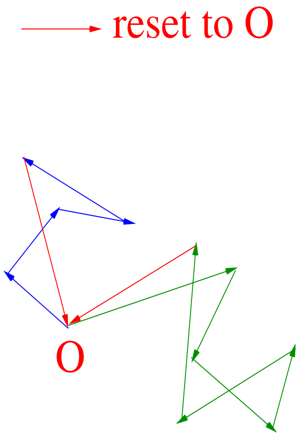

Other examples of stochastic resetting

- Searching for the global minimum in a complex energy landscape via simulated annealing

  empirical observation: Resetting to the initial configuration from time to time (and starting afresh) helps finding new pathways out of a metastable configuration

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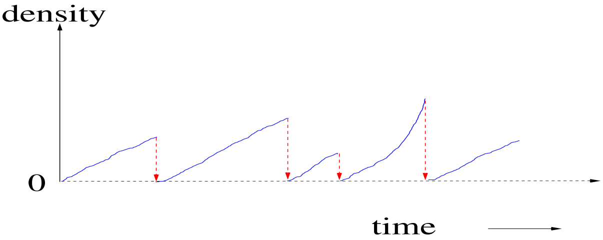
- Evolution of bacterial population: applying antibiotics from time to time resets the bacterial density to zero



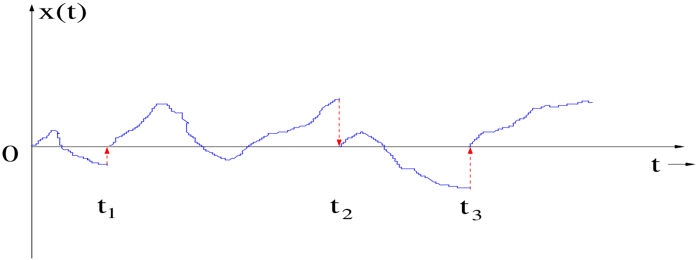
Stochastic resetting in other contexts

- Multiplicative processes ⇒ Manrubia and Zanette, 1999
- Network search ⇒ Gelenbe, 2010
- Randomized search algorithms in combinatorial optimization ⇒

Montanari and Zecchina, 2002

Janson and Peres, 2012

Stochastic Resetting

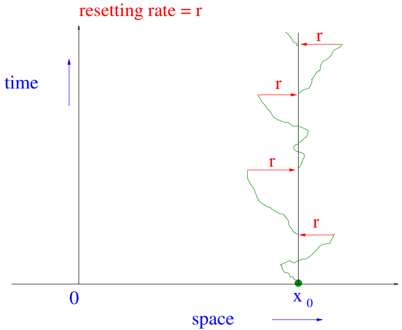


- Consider any process $x(t)$ evolving freely by its own dynamics (deterministic or stochastic) during a certain random interval of time
- At the end of this random period, the process is reset to its initial position and the its dynamics restarts afresh
- The interval of free evolution between resets is drawn independently from a distribution $p(\tau) \Longrightarrow$ renewal process
- For Poissonian resetting with a constant rate r: $p(\tau) = r\, e^{-r\tau}$

I : Diffusion with stochastic resetting

[M.R. Evans & S.M., PRL, 106, 160601 (2011)]

Diffusion with stochastic resetting: The model



Poissonian resetting

$$p(\tau) = r\, e^{-r\tau}$$

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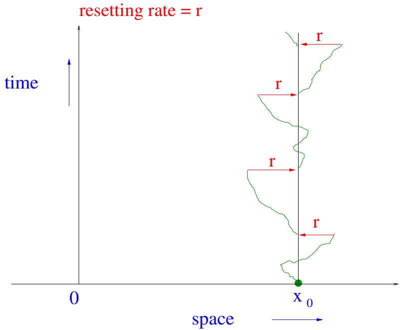

Poissonian resetting

$$p(\tau) = r\, e^{-r\tau}$$

Dynamics: In a small time interval Δt

$$x(t+\Delta t) = x_0 \quad \text{with prob. } r\Delta t \quad \text{(resetting)}$$

$$= x(t) + \eta(t)\,\Delta t \quad \text{with prob. } 1 - r\Delta t \quad \text{(diffusion)}$$

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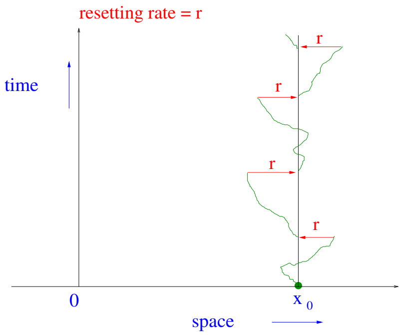

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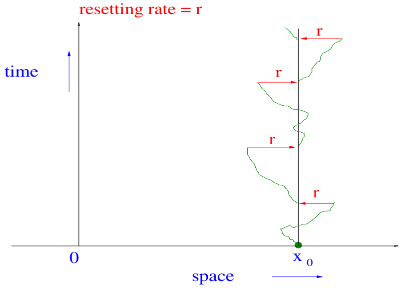
$\eta(t) \to$ Gaussian white noise: $\langle \eta(t) \rangle = 0$ and $\langle \eta(t)\eta(t') \rangle = 2\, D\, \delta(t-t')$

[M.R. Evans & S.M., PRL, 106, 160601 (2011)]

Prob. density $p_r(x,t)$ with resetting rate $r>0$



$p_r(x,t) \to$ prob. density at time t,
given $p_r(x,0)=\delta(x-x_0)$

- In the absence of resetting ($r=0$):

$$p_0(x,t)=\frac{1}{\sqrt{4\pi\, D\, t}}\ \exp[-(x-x_0)^2/4Dt]$$

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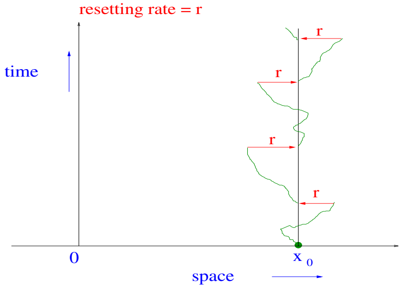

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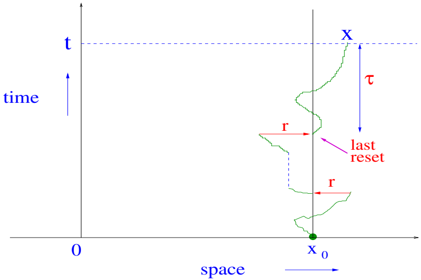
- In the presence of resetting ($r > 0$):

$$p_r(x,t) = ?$$

Renewal solution valid at all times t



$\tau \rightarrow$ time since the last resetting during which free diffusion

- $0 \le \tau \le t \rightarrow$ random variable

$$\mathrm{Prob.}[\tau|t] = r\, e^{-r\,\tau} \qquad \text{for } 0 \le \tau < t$$
$$= e^{-rt}\,\delta(\tau - t) \text{ for } \tau = t$$

(no resetting in $[0, t]$)

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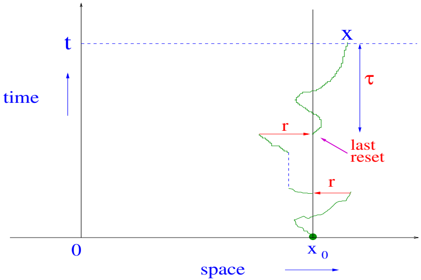

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- Renewal structure:

$$p_r(x,t) = \int_0^t d\tau\, (r\, e^{-r\,\tau})\, p_0(x,\tau) + e^{-r\,t}\, p_0(x,t)$$

$\Longrightarrow$ full exact solution at all times t

where $p_0(x,\tau) = \text{diffusion propagator} = \frac{1}{\sqrt{4\pi\, D\, \tau}}\, \exp[-(x-x_0)^2/4D\tau]$

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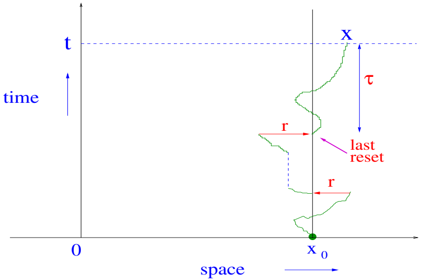

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- As $t \to \infty$, $p_r^{\rm st}(x) = r\, \int_0^\infty p_0(x,\tau)\, e^{-r\,\tau}\, d\tau = \frac{\alpha_0}{2}\, \exp[-\alpha_0\, |x - x_0|]$

where $\alpha_0 = \sqrt{r/D}$

Stationary State

Exact solution $\to$ $\boxed{p_r^{\mathrm{st}}(x) = \frac{\alpha_0}{2}\, \exp[-\alpha_0\, |x - x_0|]}$ with $\alpha_0 = \sqrt{r/D}$

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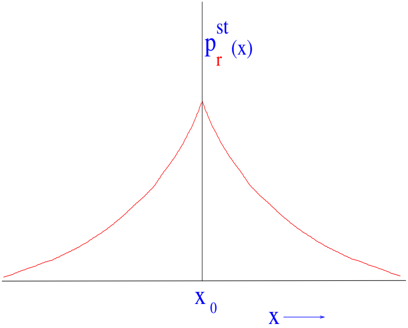


$\rightarrow$ nonequilibrium stationary state (NESS)

$\Rightarrow$ current carrying with detailed balance $\rightarrow$ violated

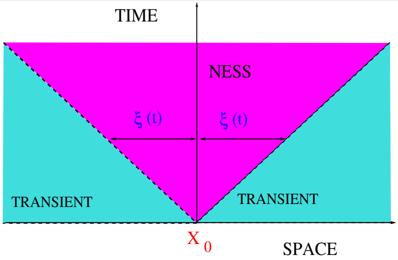
$$p_r^{\mathrm{st}}(x) = \alpha_0\, \exp[-V_{\mathrm{eff}}(x)]$$

effective potential:

$$V_{\mathrm{eff}}(x) = \alpha_0 |x - x_0|$$

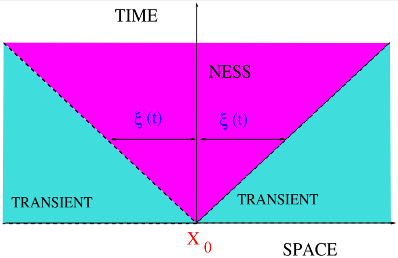
II : Unusual temporal relaxation

Dynamical phase transition



$$p_r(x,t) \sim \exp[-\alpha_0\,|x-x_0|] \qquad \text{for } |x-x_0| \le \xi(t) \quad \text{(NESS)}$$
$$\sim \exp[-r\,t - |x-x_0|^2/4Dt] \quad \text{for } |x-x_0| \ge \xi(t) \ \text{(TRANSIENT)}$$

Dynamical phase transition



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where $\alpha_0 = \sqrt{r/D}$ and $\xi(t) = \sqrt{4\,D\,r}\;t \Rightarrow$ growing length scale

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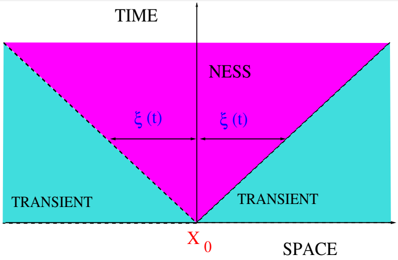

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$\Longrightarrow$ NESS gets established on larger and larger length scales

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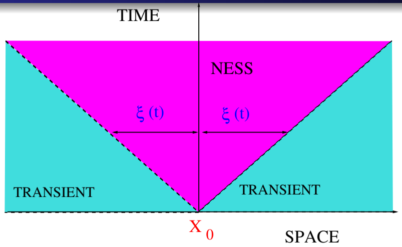
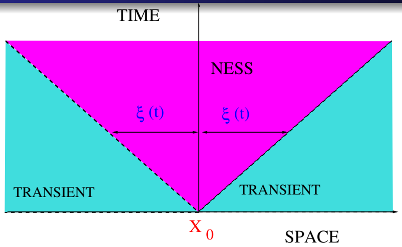


Large deviation form: $$\boxed{p_r(x,t) \sim \exp\left[-t\, I\left(\frac{|x-x_0|}{t}\right)\right]}$$

Dynamical phase transition

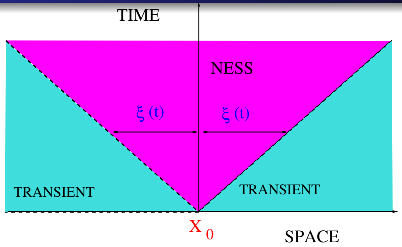


Large deviation form: $\boxed{p_r(x,t) \sim \exp\left[-t\, I\left(\frac{|x-x_0|}{t}\right)\right]}$

where the rate function

$$\begin{aligned} I(y) &= \alpha_0\, |y| && \text{for } |y| \leq y^* = \sqrt{4Dr} \\ &= r + y^2/4D && \text{for } |y| \geq y^* = \sqrt{4Dr} \end{aligned}$$

Dynamical phase transition



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\end{aligned}
$$

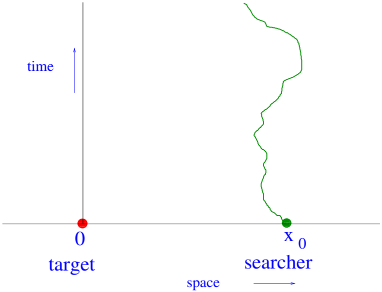
second derivative $I''(y)$ is discontinuous at $y = y^*$

$\Longrightarrow$ 2-nd order dynamical phase transition

[S.M., S. Sabhapandit, G. Schehr, PRE, 91, 052131 (2015)]

III : Target Search: First-passage properties

Search of a fixed target via pure diffusion



$Q_0(x_0, t) \to$ persistence/survival prob. of the target up to t, starting at x_0

satisfies the backward Fokker-Planck equation:

$\partial_t Q_0(x_0, t) = D\, \partial_{x_0}^2 Q_0(x_0, t)$ for $x_0 \geq 0$

with appropriate boundary/initial cond.

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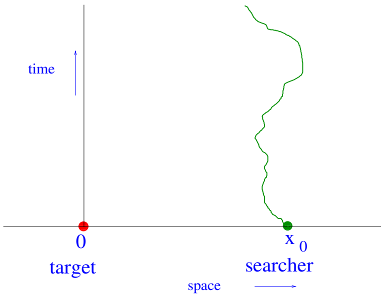

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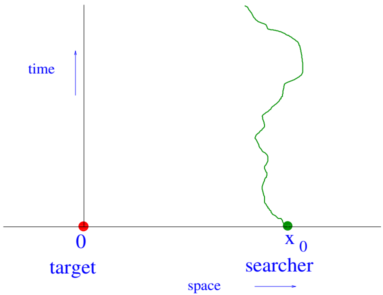

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$$F_0(x_0,t) = -\partial_t Q_0(x_0,t) = \frac{x_0}{\sqrt{4\pi D t^3}}\,\exp[-x_0^2/4Dt] \xrightarrow[t\to\infty]{} t^{-3/2}$$

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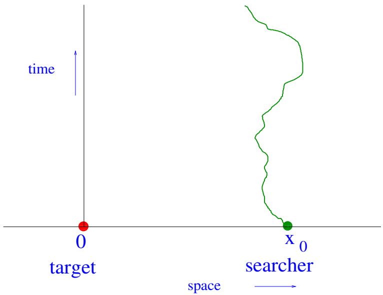

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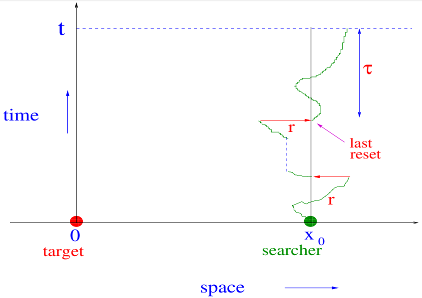
- Mean capture time $\to \bar{T} = \int_0^\infty t\, F_0(x_0, t)\, dt = \infty$

Target search via diffusion with resetting



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during which free diffusion

- $0 \le \tau \le t \to$ random variable

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$$= e^{-rt}\, \delta(\tau - t) \text{ for } \tau = t$$

(no resetting in $[0, t]$)

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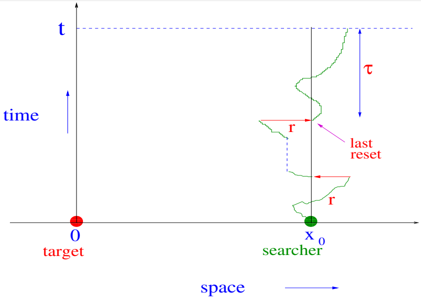

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Renewal equation for survival prob. $Q_r(x_0, t)$:

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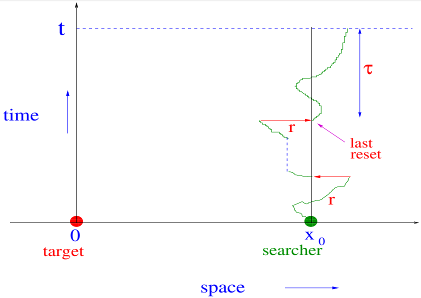

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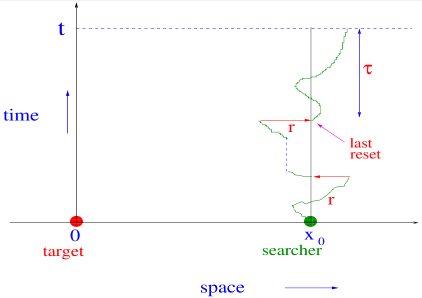
Laplace transform: $\tilde{Q}_r(x_0, s) = \int_0^\infty Q_r(x_0, t)\, e^{-s\,t}\, dt$

Target search via diffusion with resetting



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$$\Longrightarrow \qquad \tilde{Q}_r(x_0,s) = \frac{\tilde{Q}_0(x_0,s+r)}{1 - r\,\tilde{Q}_0(x_0,s+r)}$$

Exact survival probability

- Survival prob. in the presence of resetting:

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- Using, for the free diffusion ($r = 0$)

$$\tilde{Q}_0(x_0, s) = \frac{1}{s}\left[1 - e^{-\sqrt{s/D}\, x_0}\right]$$

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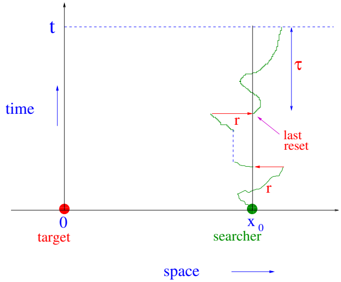
$$\Longrightarrow \qquad \tilde{Q}_r(x_0, s) = \frac{1 - \exp\left(-\sqrt{(r+s)/D}\, x_0\right)}{s + r\, \exp\left(-\sqrt{(r+s)/D}\, x_0\right)}$$

[M.R. Evans & S.M., PRL, 106, 160601 (2011)]

Mean capture/search time



Mean capture time: $\bar{T} = \int_0^\infty t\,[-\partial_t Q_r(x_0,t)]\,dt = \tilde{Q}_r(x_0, s=0)$

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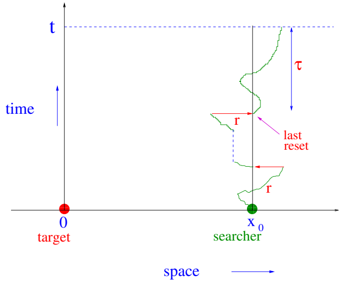

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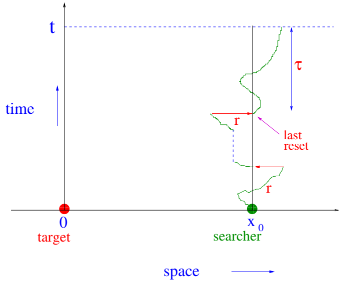

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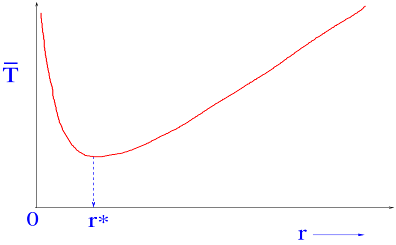
$\Longrightarrow$ mean capture time is ∞ for $r = 0$, but finite when $r > 0$

Optimal resetting rate



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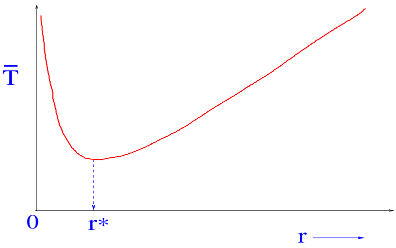

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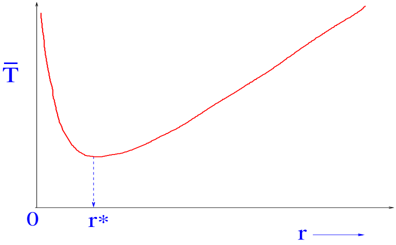

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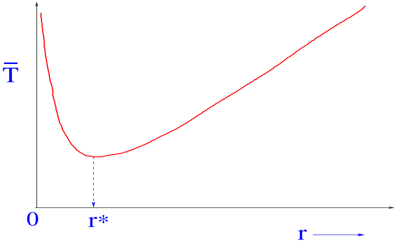

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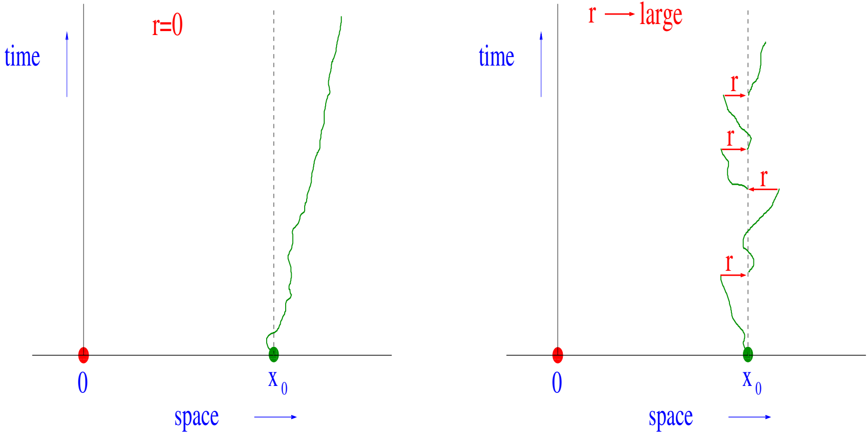
optimal resetting rate r^* is given by:

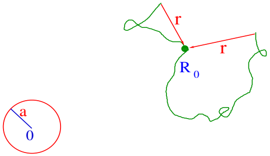
$$\boxed{r^* = \gamma^2 \frac{D}{x_0^2}} \text{ where } \boxed{\gamma - 2\left(1 - e^{-\gamma}\right) = 0} \Rightarrow \gamma = 1.59362\ldots$$

(M.R. Evans and S.M., Phys. Rev. Lett. 106, 160601 (2011))

Typical trajectories for $r \to 0$ and $r \to \infty$



Target search via diffusion with resetting in $d > 1$



stationary target of radius a at 0 in $d > 2$

searcher starts at $R_0 > a$, diffuses, and resets with rate r

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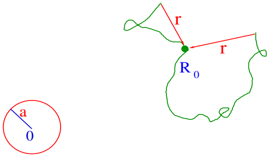

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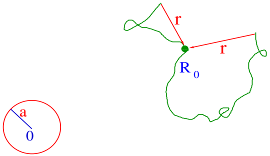

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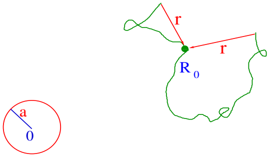

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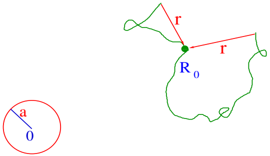

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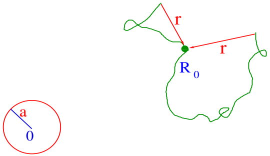

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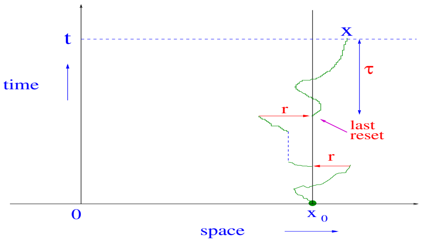
- Once again, there is an optimal r^* that minimizes $\bar{T}(r, R_0)$ in all d

[M.R. Evans and S.M., J. Phys. A: Math. Theo. 47, 285001 (2014)]

IV : **Arbitrary process with resetting**

Renewal solution for an arbitrary process



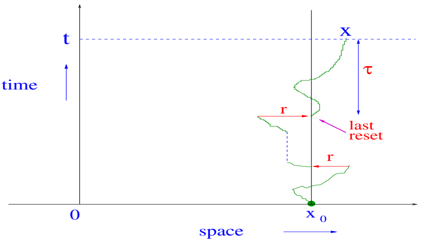
$\tau \to$ time since the last resetting during which free evolution

- $0 \le \tau \le t \to$ random variable

$$\mathrm{Prob.}[\tau|t] = r\, e^{-r\,\tau} \quad \text{for } 0 \le \tau < t$$
$$= e^{-rt}\,\delta(\tau - t) \text{ for } \tau = t$$

(no resetting in $[0, t]$)

Renewal solution for an arbitrary process



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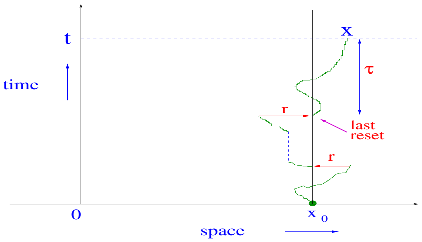
- Renewal structure:

$$p_r(x,t) = \int_0^t d\tau\,(r\,e^{-r\,\tau})\,p_0(x,\tau) + e^{-r\,t}\,p_0(x,t)$$

$\Longrightarrow$ full exact solution at all times t

where $p_0(x,t) \longrightarrow$ bare propagator

Renewal solution for an arbitrary process



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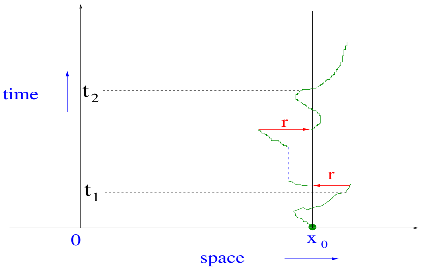
- As $t \to \infty$

$$p_r^{\mathrm{st}}(x) = r \int_0^\infty p_0(x,\tau)\, e^{-r\,\tau}\, d\tau$$

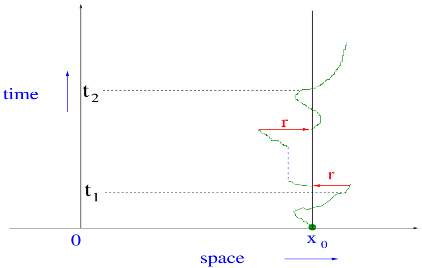
Auto-correlation function with resetting



Auto-correlation function:

$C_r(t_1, t_2) = \langle X_r(t_1) X_r(t_2) \rangle - \langle X_r(t_1) \rangle \langle X_r(t_2) \rangle$

Auto-correlation function with resetting



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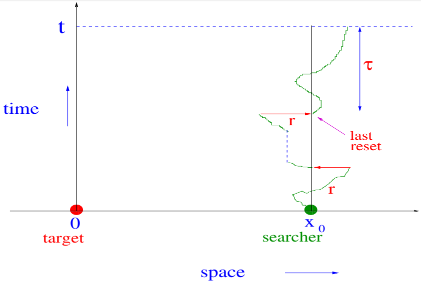
Exploiting the renewal structure: for $t_1 \leq t_2$

$$\boxed{C_r(t_1,t_2) = e^{-r(t_2-t_1)}\left[r\int_0^{t_1} d\tau\, e^{-r\tau}\, C_0(\tau, t_2-t_1+\tau) + e^{-rt_1} C_0(t_1,t_2)\right]}$$

where $C_0(t_1,t_2) \longrightarrow$ bare correlator (in the absence of resetting)

[S.M. & G. Oshanin, J. Phys. A: Math. Theo. 51, 435001 (2018)]

Survival/first-passage prob. with resetting



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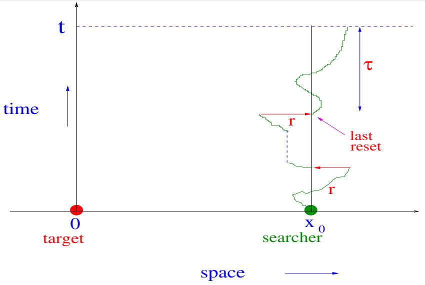

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Renewal equation for survival prob. $Q_r(x_0, t)$:

$$Q_r(x_0,t) = \int_0^t d\tau\,(r\,e^{-r\,\tau})\,Q_0(x_0,\tau)\,Q_r(x_0,t-\tau) + e^{-r\,t}\,Q_0(x_0,t)$$

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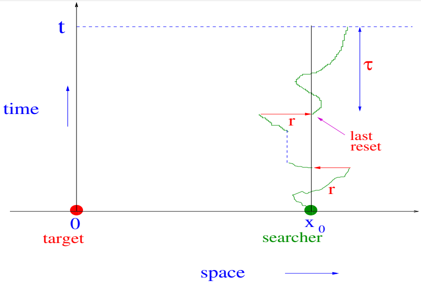

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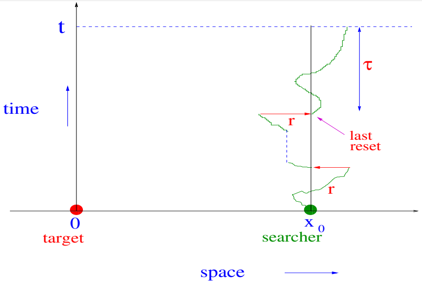

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$$\Longrightarrow \qquad \tilde{Q}_r(x_0,s) = \frac{\tilde{Q}_0(x_0,s+r)}{1 - r\,\tilde{Q}_0(x_0,s+r)}$$

Various generalisations of stochastic resetting

Over the last few years, effects of stochastic resetting have been extensively studied in many different contexts:

- Enzymatic reactions in biology (Michaelis-Menten reaction)
- Diffusion in a confining potential/box
- Lévy flights, Lévy walks, fractional BM with resetting
- Space-dependent resetting rate $r(x)$
- Power-law distributed time interval between successive resets
- Search via nonequilibrium reset dynamics vs. equilibrium dynamics
- Resetting dynamics of extended systems (e.g. fluctuating interfaces)
- Properties of functionals of reset processes
- Memory dependent reset
- Quantum dynamics with reset
- Active run-and-tumble dynamics with reset

... $\Longrightarrow$ a long list **!** (many people have made important contributions **!**)

Experiments?

Theory of resetting $\Longrightarrow$ rapidly developing

Experiments?

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How about experiments?

Experiments?

Theory of resetting $\Longrightarrow$ rapidly developing

How about experiments?

Ongoing experiments on target search via diffusion with resetting using optical traps set-up (in collaboration with the group of S. Ciliberto at ENS-Lyon).

Summary and Conclusion

- A brief and partial overview of Stochastic Resetting
  $\Longrightarrow$ a rapidly evolving field of research

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- Search of a stationary target via diffusion+resetting $\rightarrow$ efficient

  Mean search time $\bar{T}(r)$ has a minimum at an optimal resetting rate r^* in all dimensions

  ... many more recent studies on the dependence of r^* on system parameters $\longrightarrow$ phase transitions

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Resetting $\rightarrow$ rich and interesting static and dynamic phenomena

Collaborators

- D. Boyer, A. Falcon-Cortes (UNAM, Mexico)
- S. Ciliberto & group (ENS-Lyon, France)
- F. den Hollander (Leiden University, The Netherlands)
- M. R. Evans, J. Whitehouse (Edinburgh University, UK)
- L. Giuggioli (Bristol University, UK)
- S. Gupta (Belur University, Kolkata, India)
- L. Kusmierz (Inst. of Phys., Krakow, Poland $\rightarrow$ Riken Center, Japan)
- M. Magoni (LPTMS, Orsay, France)
- K. Mallick (IPHT, Saclay, France)
- J. M. Meylahn, H. Touchette (Stellenbosch University, South Africa)
- B. Mukherjee, K. Sengupta (IACS, Kolkata, India)
- G. Oshanin (Sorbonne Université, Paris, France)
- S. Sabhapandit (RRI, Bangalore, India)
- G. Schehr (LPTMS, Orsay, France)

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