Introduction to the quantum first detected passage time problem

Eli Barkai

Friedman, Kessler, Mualem, Thiel

PRE 95, 032141 (2017)

PRL 120, 040502 (2018)

arXiv:1906.08108 [quant-ph] (2019)

Outline

Consider unitary evolution pierced by repeated measurements.

The quantum version of the first passage time (Smoluchowski).

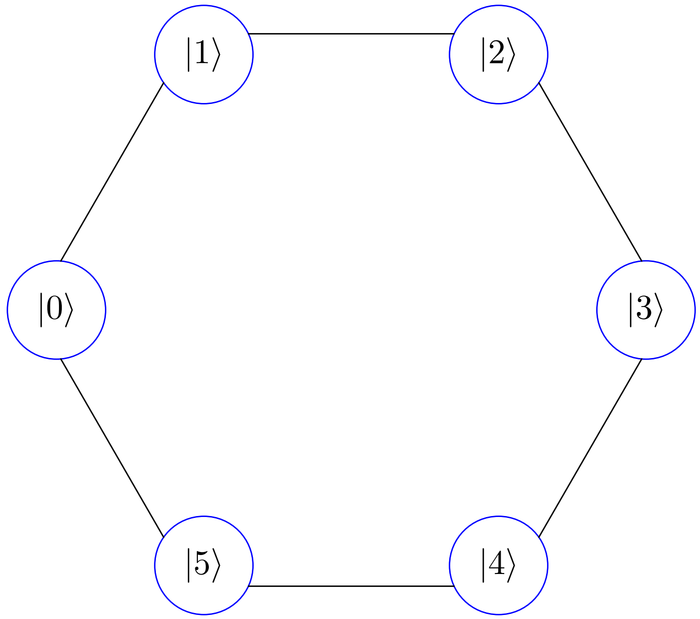
Quantum renewal equation.

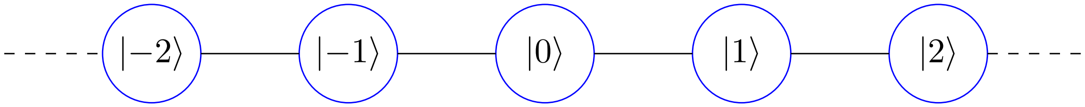
Interference sub-spaces: the dark and bright.

Uncertainty principle and symmetry bounds.

Quantization of the return time, and more.

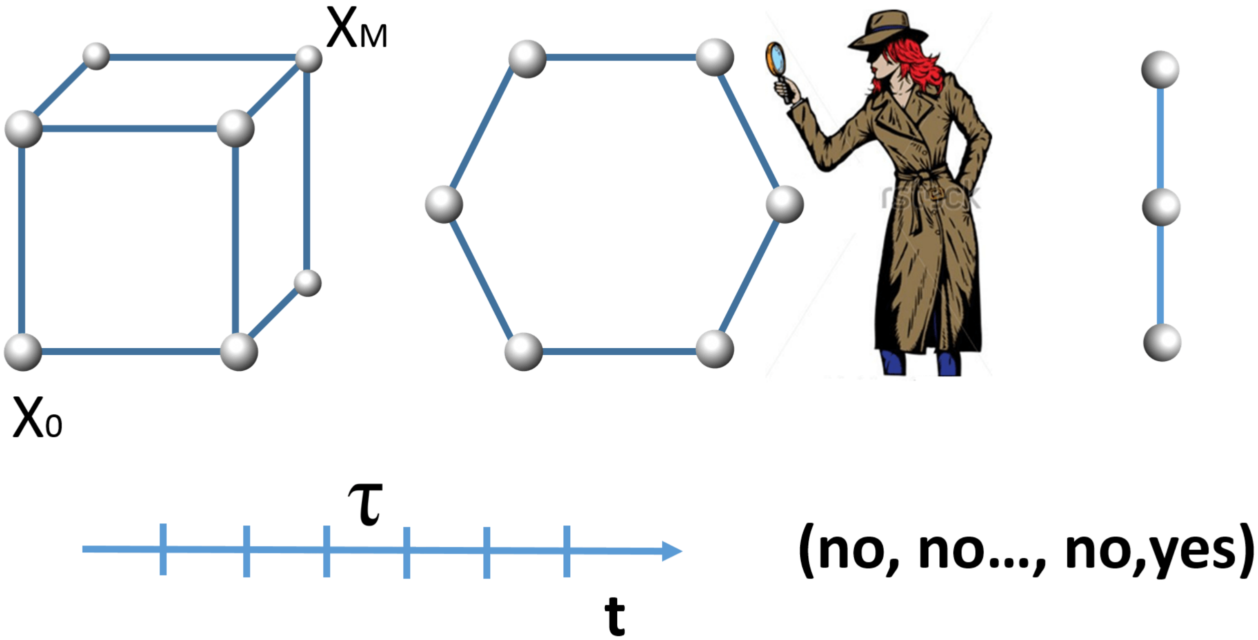
Quantum Walks, $U = \exp(-iH\tau)$





$$H = -\gamma \sum_i \left(|i\rangle\langle i+1| + |i+1\rangle\langle i| \right)$$

Measurement protocol



Quantum computing: is the quantum search superior to the classical random walk? How to choose τ? optimal sampling?

Ambainis et al (2001), Krovi and Brun (2008), Grunbaum et al (2013) $[X_0 = X_M]$

First Detection Time: Definition

So, we consider a single quantum particle evolving under a Hermitian Hamiltonian H.

Every τ seconds, we make a measurement asking whether or not the particle is detected at x_M.

If yes, the game stops, and this gives the detection time.

If no, life goes on and τ seconds later we measure again.....

We get the string (no, no, ... yes) and in the n-th entry a yes.

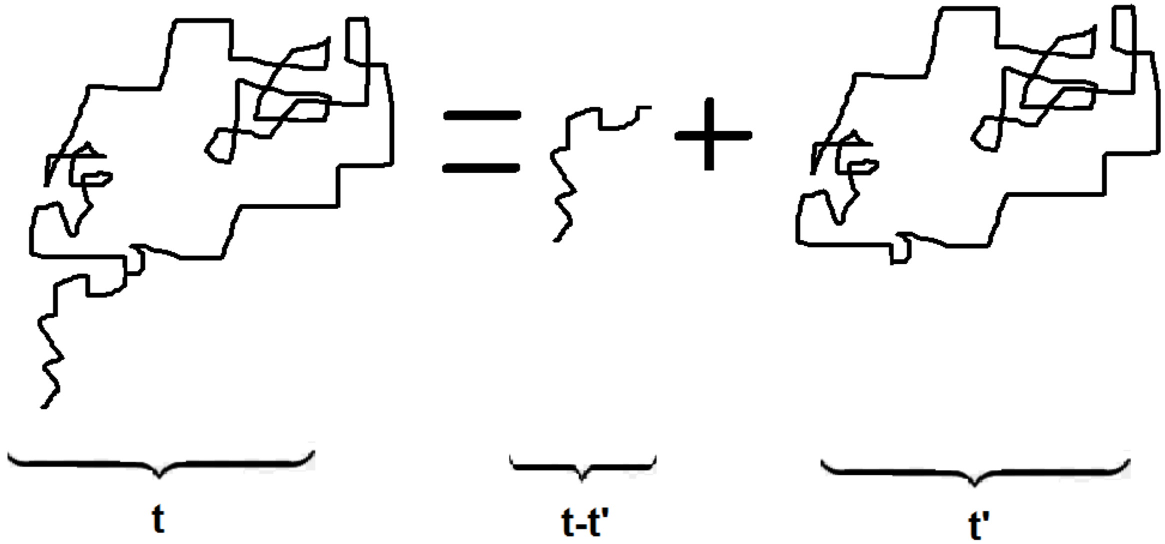
$n\tau$ is the random first detection event.

What is the distribution of n? Can we find $\langle n \rangle$?
will we detect the particle?

What is non-trivial is that the measurement process "collapses" the wave-function, setting $\psi(x_M) = 0$.

- The rest of the wave-function is unchanged, except for normalization
- In operator language, we "project out" the x_M component of the state.

Classical First Passage



Particles arriving at x at time t, first arrived at x some earlier time $t - t'$ and returned there after t' additional steps.

Quantum Renewal Equation

- ϕ_n amplitude of first detection probability (Dhar).
- $F_n = |\phi_n|^2$ Prob. of first detection in the n-th attempt.
- $P_{det} = \sum_{n=1} F_n$ can be less than one even on finite graphs.

$$\boxed{\langle x_M|U(n\tau)|\psi_{in}\rangle = \sum_{j=1}^{n}\langle x_M|U\left[(n-j)\tau\right]|x_M\rangle\phi_j}$$

- For example: $\phi_1 = \langle x_M|U(\tau)|\psi_{in}\rangle$

$$\phi_2 = \langle x_M|U\left(1 - |x_M\rangle\langle x_M|\right)U|\psi_{in}\rangle.$$

Friedmann, DK, EB **PRE** (2017)

Generating Function for ϕ_n

Generating Function:

$$\hat{\phi}(z) \equiv \sum_{n>0} z^n \phi_n$$

A single site initial condition $|\psi_{in}\rangle = |x_M\rangle$.

$$\hat{\phi}(z) = \frac{\langle x_M|\hat{U}(z)|x_M\rangle}{1 + \langle x_M|\hat{U}(z)|x_M\rangle} = 1 - \frac{1}{1 + \langle x_M|\hat{U}(z)|x_M\rangle}$$

But

$$1+\langle x_M|\hat{U}(z)|x_M\rangle = \sum_{n\geq 0} z^n \langle x_M|e^{-iH\tau n}|x_M\rangle = \sum_{n\geq 0} z^n \langle x_M|\psi_f(n\tau)\rangle \equiv \langle x_M|\psi_f(z;x_M)\rangle$$

and $|\psi_f(z;x_0)\rangle$ is the generating function **of the measurement-free state** starting at x_M at $t=0$!

$$\hat{\phi}(z) = 1 - \frac{1}{\langle x_M|\psi_f(z;x_M)\rangle}$$

Take home message

The measurement free state function, $|\psi_{free}\rangle$, a.k.a wave function free of measurement, gives the amplitude of first detection event on a site.

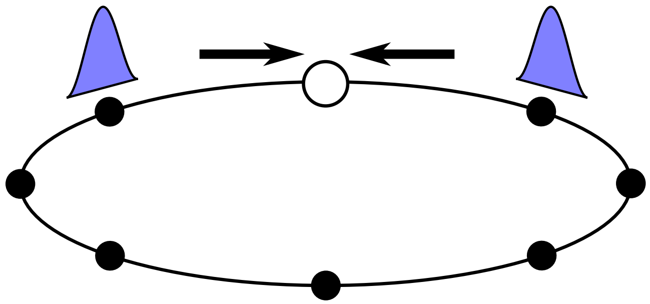
Operationally: Solve the Schrödinger equation, find the generating function of measurement free state function, perform inverse z transform, and get the detection time statistics.

Some tricks

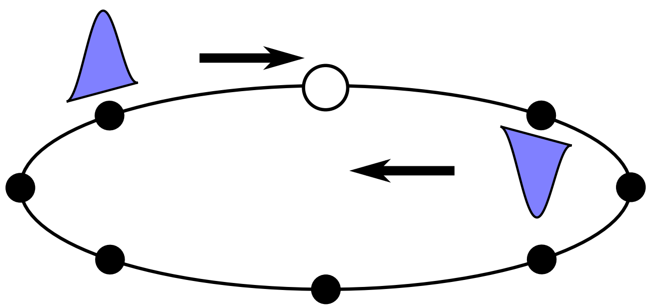
$$P_{det} = \sum_{n=1}^{\infty} F_n = \sum_{n=1}^{\infty} |\phi_n|^2 = \frac{1}{2\pi}\int_0^{2\pi} \sum_{k=1}^{\infty} \phi_k e^{i\theta k} \sum_{l=1}^{\infty} \phi_l^* e^{-i\theta l} d\theta = \frac{1}{2\pi}\int_0^{2\pi} |\hat{\phi}(e^{i\theta})|^2 d\theta.$$

$$\langle n \rangle = \sum_{n=1}^{\infty} n F_n = \frac{1}{2\pi}\int_0^{2\pi} \left[\hat{\phi}\left(e^{i\theta}\right)\right]^* \left(-i\frac{\partial}{\partial\theta}\right) \hat{\phi}(e^{i\theta}) d\theta.$$

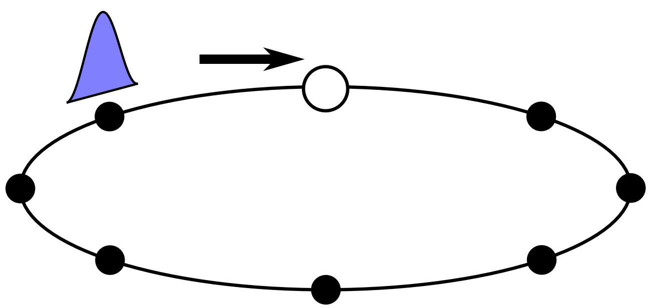
Friedmann, DK, EB **PRE** (2017)



Constructive interference

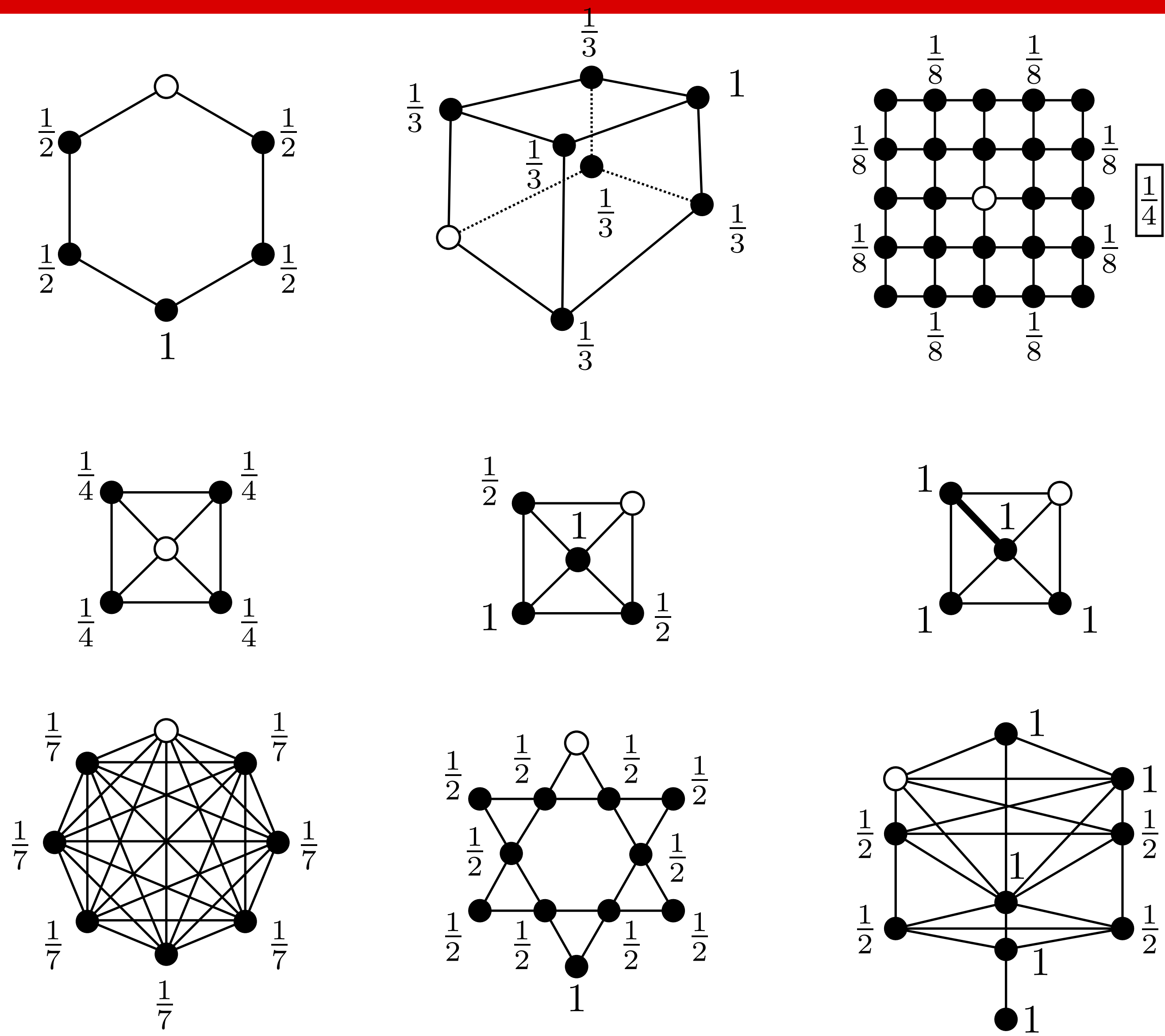


Destructive interference



$$P_{\mathrm{det}} \leq \frac{1}{2}$$

Detection probability P_{det}



Detection probability

- Start in an initial state

$$|\psi_{in}\rangle = \frac{1}{\sqrt{2}}\left(|r_i\rangle + e^{i\delta}|r_j\rangle\right)$$

- From quantum renewal equation detection amplitudes sum up

$$\phi_n^{\psi_{in}\to r_d} = \frac{1}{\sqrt{2}}\left(\phi_n^{r_i\to r_d} + e^{i\delta}\phi_n^{r_j\to r_d}\right).$$

- If states $|r_j\rangle$ and $|r_j\rangle$ are equivalent with respect to detection. Then:

$$F_n^{\psi_{in}\to r_d} = (1+\cos\delta)F_n^{r_i\to r_d}$$

- Since $\sum_{n=1}^{\infty} F_n^{\psi_{in}\to r_d} \le 1$ we have

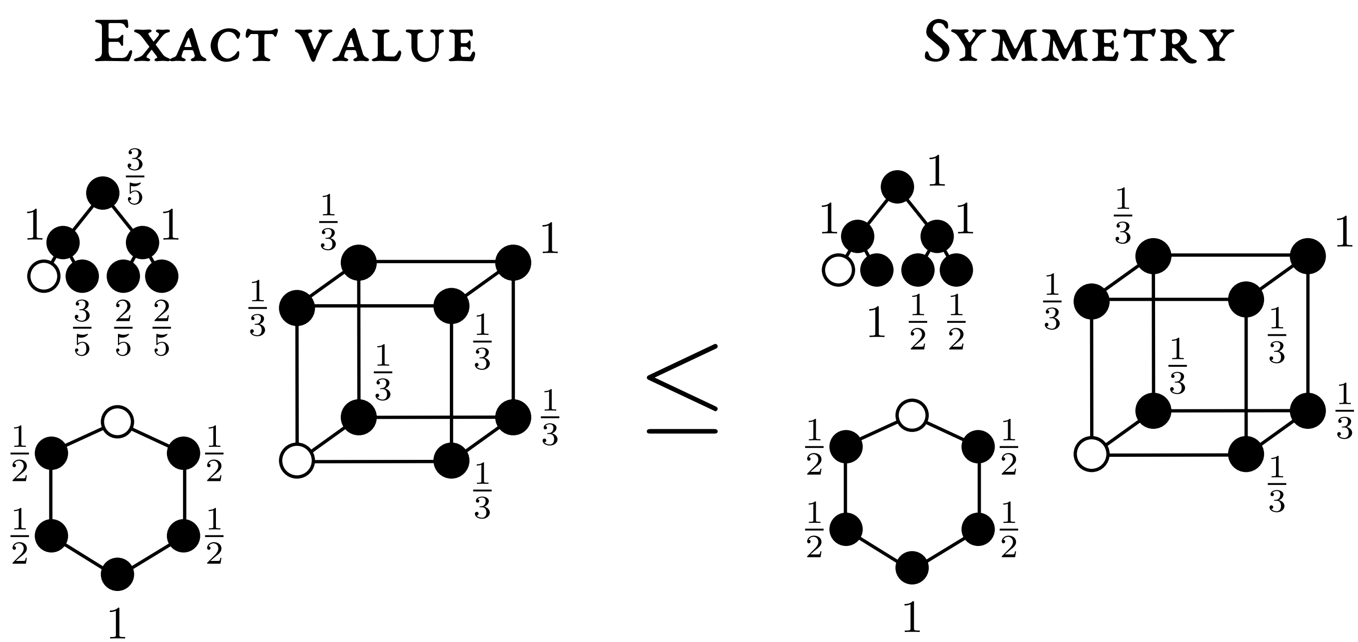
$$(1+\cos\delta)\sum_{n=1}^{\infty} F_n^{r_i\to r_d} \le 1.$$

Choose $\delta = 0$ so $P_{det}(r_i \to r_d) \le \frac{1}{2}$.

- More generally, consider a graph with ν initial sites all equivalent with respect to the detection. We have

$$\boxed{P_{det}^{r_i \to r_d} \leq 1/\nu}$$

- Quantum detection is less efficient if compared with classical random walks.



- More generally $P_{det} = \sum_l |\langle \text{bright}_l | \psi_{in} \rangle|^2$ and after the classification of the bright states we find

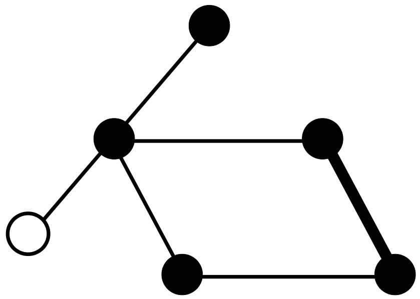
$$\boxed{P_{det} = \sum_l{}' \frac{|\sum_{m=1}^{g_l} \langle x_M | E_{l,m} \rangle \langle E_{l,m} | \psi_{in} \rangle|^2}{\sum_{m=1}^{g_l} |\langle x_M | E_{l,m} \rangle|^2}.}$$

- In the case of no degeneracy (removal of symmetry) detection is unity.
- Disorder is good for efficient search [Plenio - light harvesting systems].
- P_{det} is τ independent.
- Valid beyond the stroboscopic protocol.

Thiel, Mualem, DK, EB (2019).

Hilbert space under repeated measurement

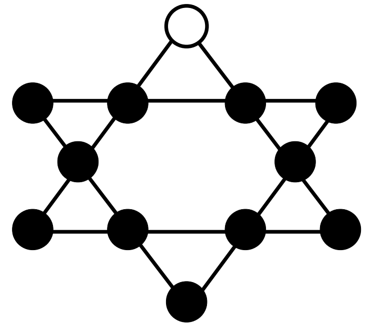
Zeno



$\tau \to 0$

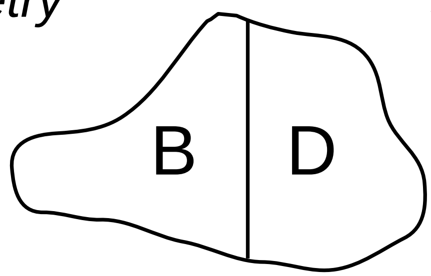
no symmetry

Interference



$\tau \neq 0$

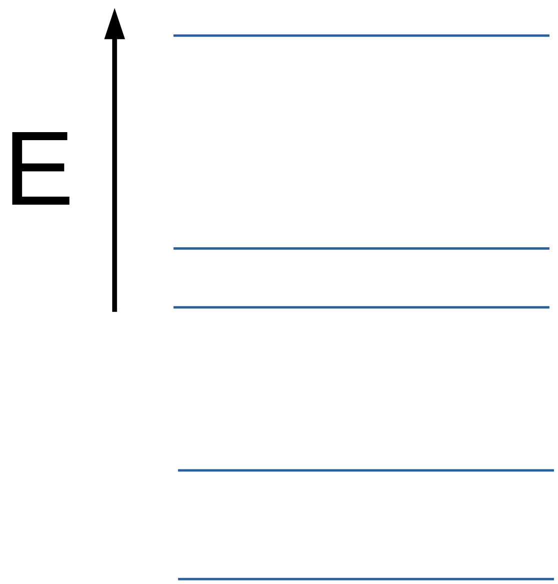
symmetry



Dark and Bright

- A bright/dark state is detected with probability one/zero.
- The Hilbert space can be decomposed into bright and dark subspaces.
- For rapid measurement this is related to the Zeno effect.
- Dark states are related to degeneracy and hence to symmetry.
- For degenerate states: $|\psi\rangle = N\left(\langle x_M|E_2\rangle|E_1\rangle - \langle x_M|E_1\rangle|E_2\rangle\right)$ is dark.
- Also a non degenerate energy level can be dark if $\langle x_M|E_3\rangle = 0$.
- Alan Turing, Sudarshan, Misra, Plenio, Facchi, Paasazio, Caruso, Krovi ....

Sketch- Dark Bright States



g_l	# Bright	# Dark
1	1	0
5	1	4
8	1	7
7	1	6
1	1	0

Start $|\psi\rangle = N\left(\langle x_M|E_2\rangle|E_1\rangle - \langle x_M|E_1\rangle|E_2\rangle\right)$.
Find other orthogonal dark states total $g_l - 1$.
From here get the total detection probability P_{det}.

Uncertainty Relation

$$\boxed{\Delta P \mathsf{Var}(H)_M \geq |\langle x_M| \, [H, D] \, |\psi_{in}\rangle|^2}$$

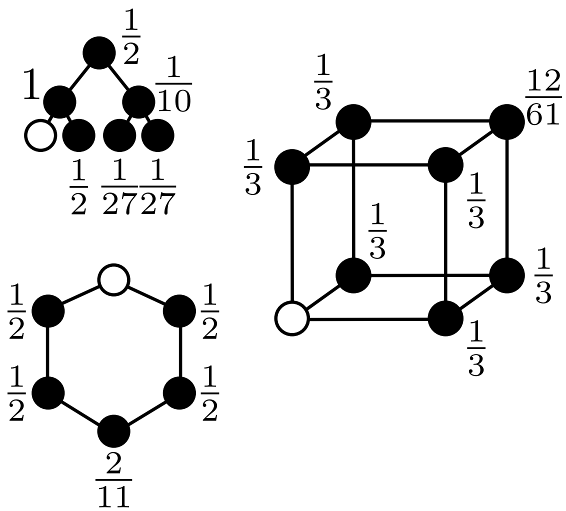
- ΔP is the deviation of the total detection probability from the initial probability of detection.

$$\Delta P = P_{det} - |\langle \psi_{in} | x_M \rangle|^2.$$

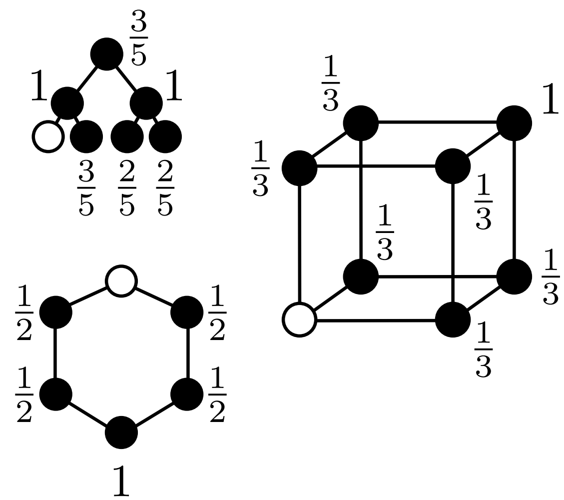
- $\mathsf{Var}(H)_M$ is the variance of the energy in the detected state.
- $D = |x_M\rangle\langle x_M|$ is the measurement projector.
- On the RHS we connect the initial and final states.

Uncertainty

Exact value



$\leq$



Into the bright space

- Let $|\beta\rangle$ be a bright state. Then $f(H)|\beta\rangle$ is also bright.
- To see this use $\langle E^D|f(H)|\beta\rangle = 0$.
- $|x_M\rangle, H|x_M\rangle....H^k|x_M\rangle...$ are bright and so are:

$$|B_1\rangle = |x_M\rangle \text{ and } |B_2\rangle = N\left(|x_M\rangle - \frac{H|x_M\rangle}{\langle x_M|H|x_M\rangle}\right).$$

- From normalization, you get the fluctuations of H in the detected state. Using

$$P_{det} \geq |\langle B_1|\psi_{in}\rangle|^2 + |\langle B_2|\psi_{in}\rangle|^2$$

we get the uncertainty principle. Notice it is τ independent.

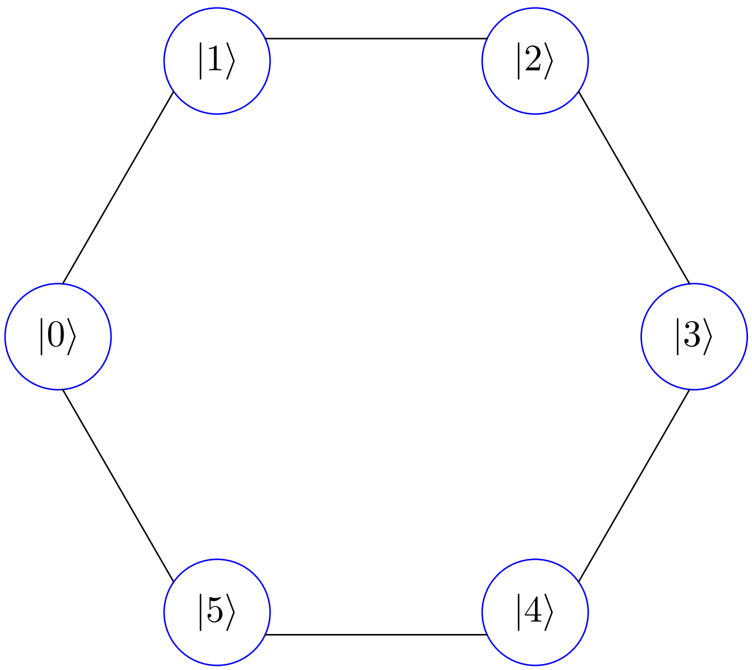
An $L=6$ Ring

Easy to compute $\hat{\phi}(z)$ for an $L=6$ ring with our hopping Hamiltonian.

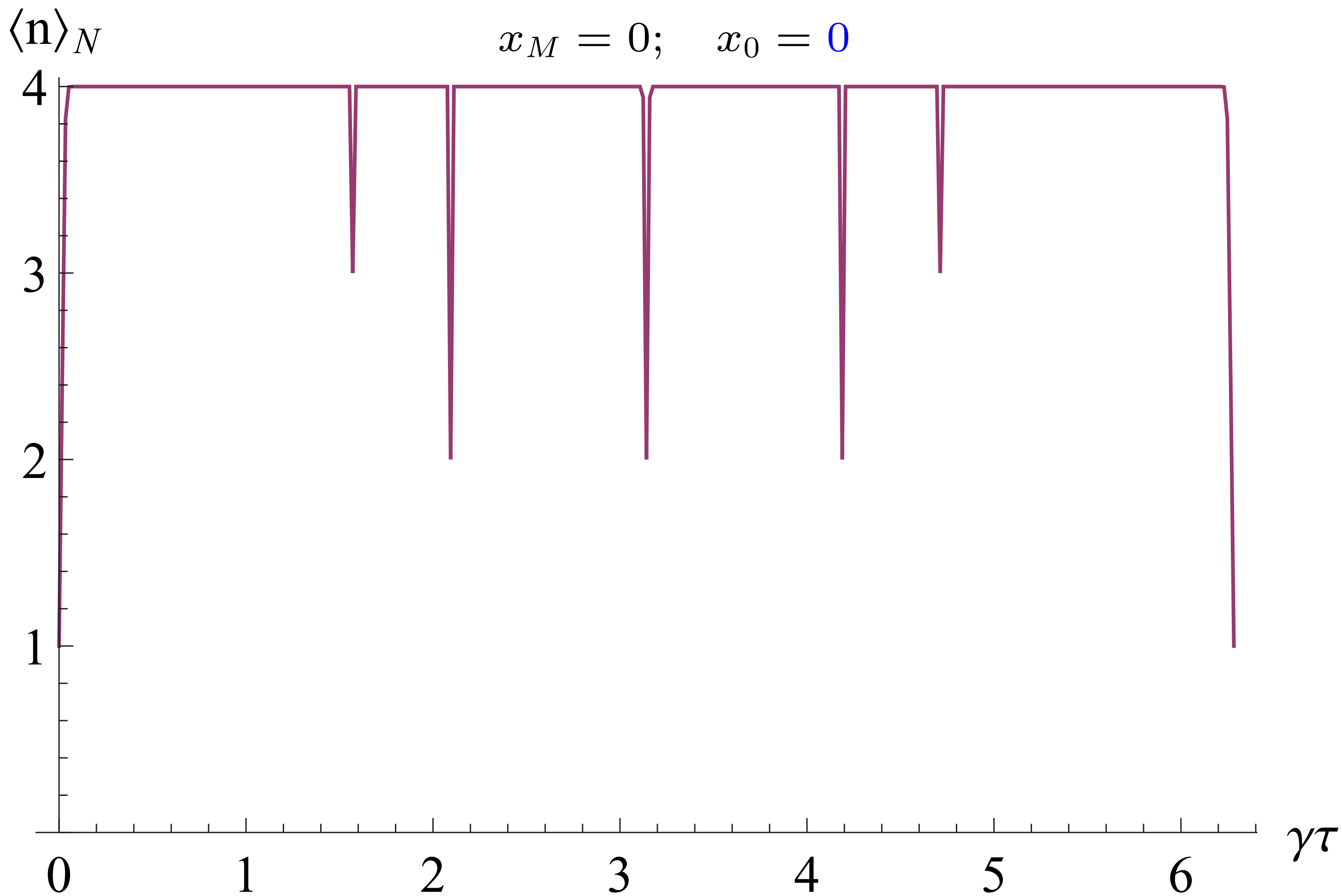
- For example, for $x_0 = x_M$:

$$\hat{\phi}(z) = \frac{\Re\left[\frac{1}{z^{-1}e^{2i\gamma\tau}-1}\right] + 2\Re\left[\frac{1}{z^{-1}e^{i\gamma\tau}-1}\right]}{3+\Re\left[\frac{1}{z^{-1}e^{2i\gamma\tau}-1}\right] + 2\Re\left[\frac{1}{z^{-1}e^{i\gamma\tau}-1}\right]}$$

Going back to ϕ_n is a bit more complicated.

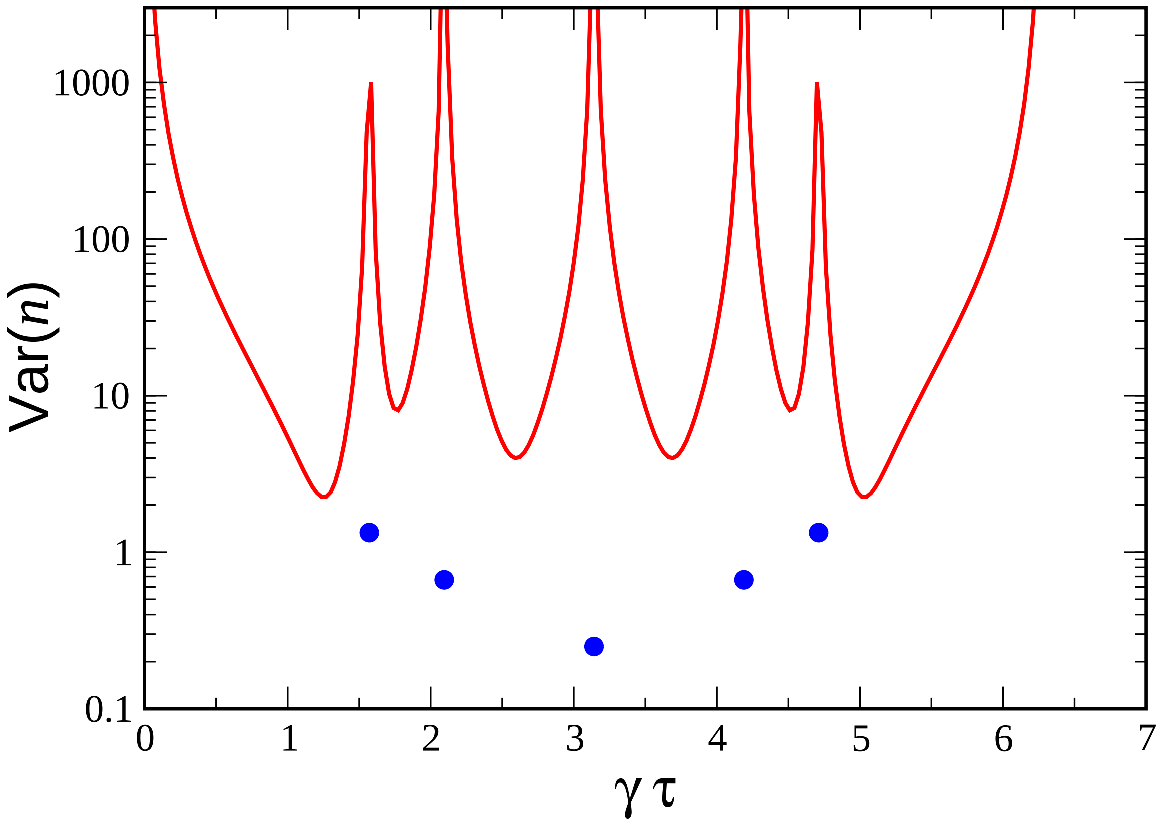


$\langle n \rangle$ an integer

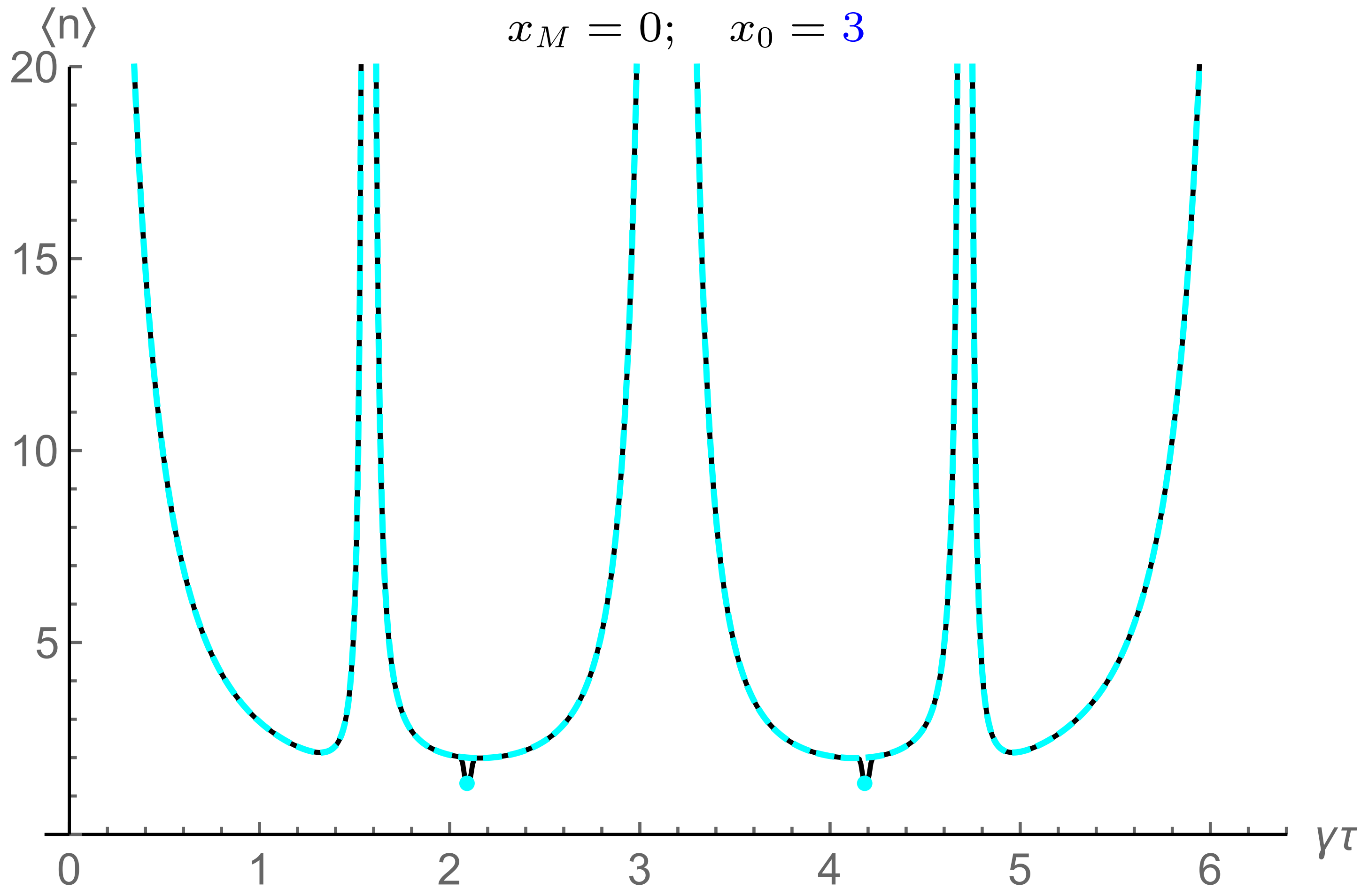


- $\langle n \rangle$ is quantized! Grünbaum, Velázquez, Werner *Comm. Math. Phys.* (2013).
- Var$[n]$ diverges at the exceptional points.

Variance



$L = 6$ Ring: the averaged first detection attempt



For $x_M = 0$, $x_0 = \{1, 2, 4, 5\}$:

- For all but some special sampling times the probability of being detected is $1/2$. **Half dark states.**

Ring of size L

- If $X_m = X_0$ the particle is detected with probability one.
- For the same initial condition and besides exceptional sampling times

$$\langle n \rangle = \begin{cases} \frac{L+2}{2} & \text{L is even} \\ \frac{L+1}{2} & \text{L is Odd} \end{cases}$$

- Exceptional sampling times

$$\boxed{\Delta E \tau = 2\pi k}$$

$$\Delta E = E_i - E_j > 0$$

Infinite Lattice in One Dimension $\gamma = 1$

The computation of $\hat{\phi}(z)$ is easy. For example for $x_0 = x_M = 0$:

$$\hat{\phi}(z) = 1 - \frac{1}{\sum_n z^n J_0(2\tau n)}$$

For large $n >> X_0/\tau$,

$$\boxed{F_n \approx \frac{4\tau r^2(X_0,\tau)}{\pi n^3} \cos^2\left[2\tau n - \beta(X_0,\tau)\right]}$$

- For $X_0 = 0$, $r = 1$, $\beta = -\pi/4$ independent of τ.
- For $X_0 >> 1$, or $\tau << 1$

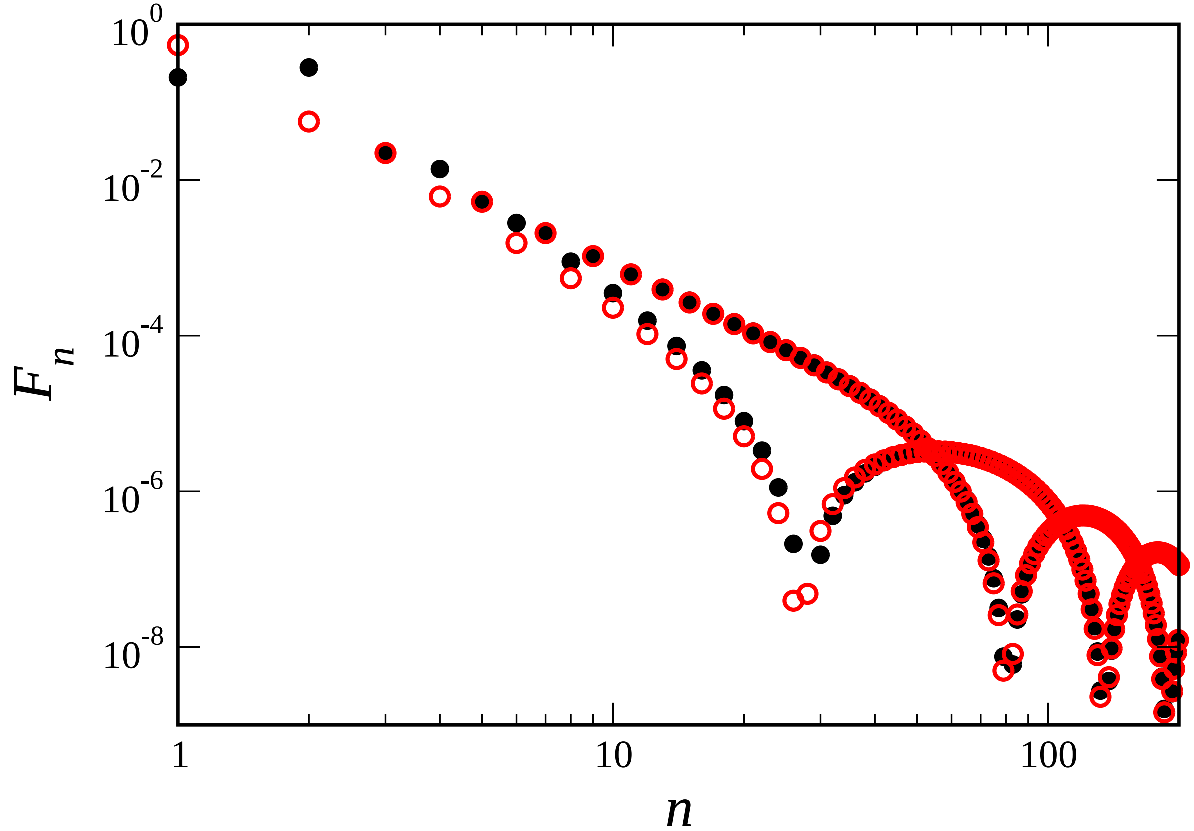
$$\boxed{F_n \sim \frac{(X_0)^2}{\pi\tau} \frac{\cos^2(2n\tau - \pi/4)}{n^3}.}$$

- For Classical Diffusion $F_n \sim n^{-3/2}$.

Thiel, EB, Kessler **PRL** (2018)

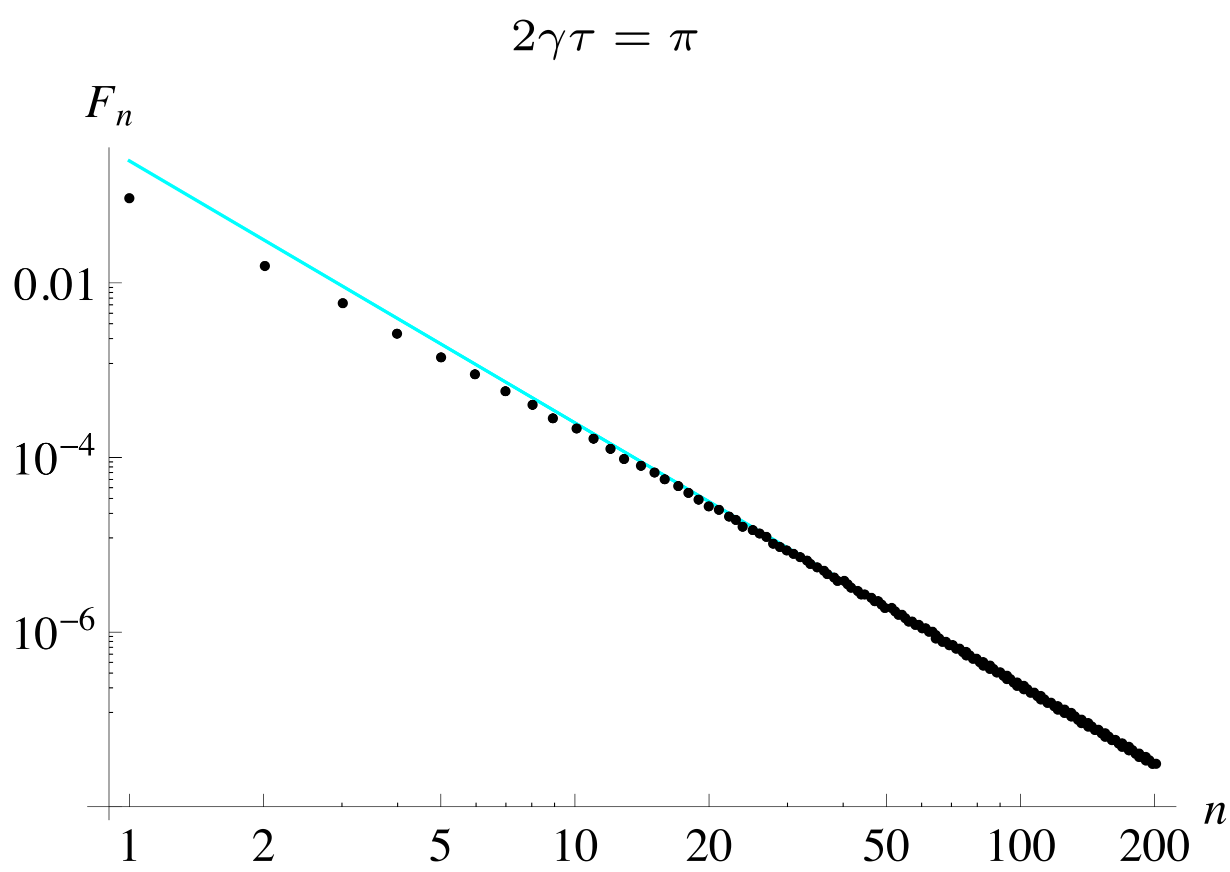
Infinite Lattice

$\gamma\tau = 0.8$



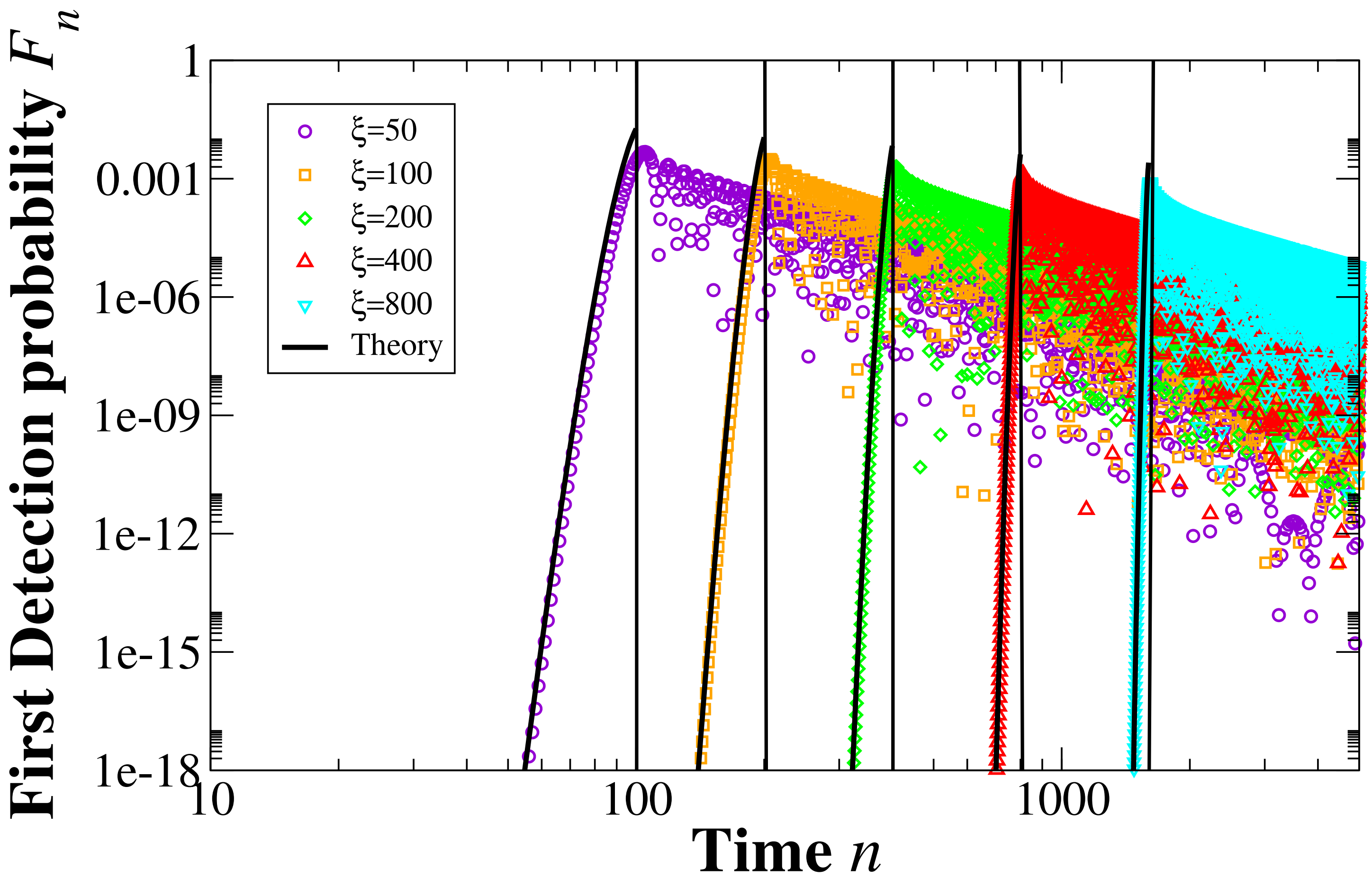
Filled Dots = Exact, Circles=Asymptotic

Critical Sampling the Infinite Lattice



- For $2\gamma\tau \to \pi$ we have $F_n \sim n^{-3}$ while at critical sampling $F_n \sim n^{-3}/4$.
- $\Delta E\tau = 2\pi$ and here $\Delta E = 4\gamma$ is the width of the energy band.

First Detection on a line



Thiel, EB, Kessler **PRL** (2018)

The incidence time

- The time the wave packet hits the detector for the first time

$$n_{inc}\tau = \frac{\Delta X}{v_g}.$$

Here $\Delta X = |X_0 - X_M|$, $v_g = max|E_k'| = 2$ ($\gamma = 1$, $E_k = 2\cos(k)$)

- For $n < N_{inc}$ we have nearly zero probability of detection.
- More precisely, for small n, with $X_M = 0$

$$F_n \sim \frac{1}{2\pi\Delta X}\left(\frac{en\tau}{\Delta X}\right)^{2\Delta X}$$

Final Remarks

Studied First Detection Time for Stroboscopic measurements.

Yields a direct analog of classical renewal equation with the first detection amplitude playing the role of the first-return probability.

Classical = Random Walk Theory and Newtonian mechanics.

Deviations from the latter, are quantified with uncertainty principle.

Resonance conditions yield discontinuities, divergences in the first-detection statistics.

Even on small graphs detection probability is not unity (provided there is no disorder).

Dark states are classified and with them we get the detection probability.

Ref. Thanks

Friedman, Kessler, EB *Quantum walks: the first detected passage time problem* **Phys. Rev. E. 95**, 032141 (2017).

Thiel, EB, and Kessler *First detected arrival of a quantum walker on an infinite line* **Phys. Rev. Lett. 120**, 040502 (2018).

Thiel, Mualem, Kessler EB *Uncertainty and symmetry bounds for the total detection probability of quantum walks* arXiv:1906.08108 [quant-ph]

Ruoyu Yin, Ziegler, Thiel, EB *Large fluctuations of the quantum first return time* arXiv:1903.03394 [cond-mat.stat-mech]

How you track matters

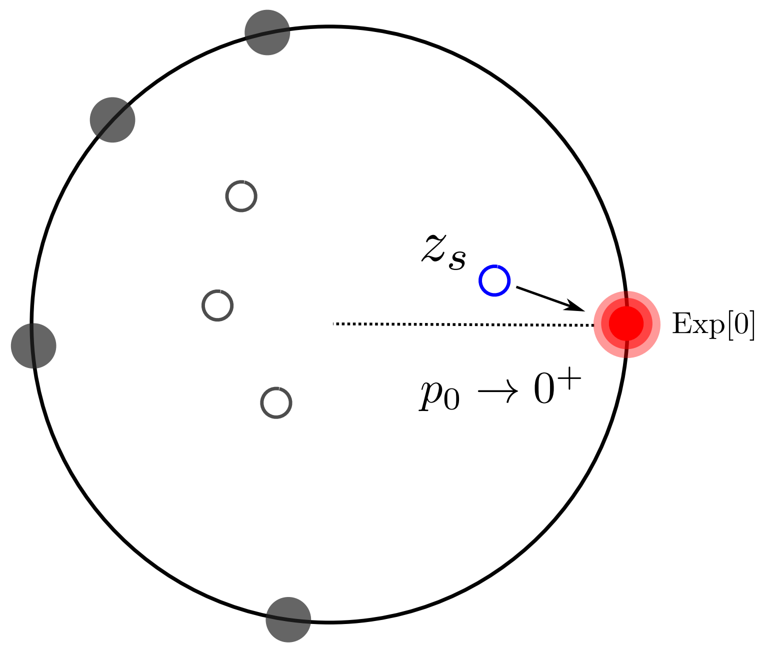
So far the measurements were local, yielding a string $no, no, ...yes$.

Consider a graph and a stroboscopic measurement of X, this yields the string $1, 42, 32, 15, ...0$ and we stop when the node 0 is visited for first time.

Also here $\langle n \rangle$ is quantized, but it is not the same as the local detection protocol (work with Avihai Didi).

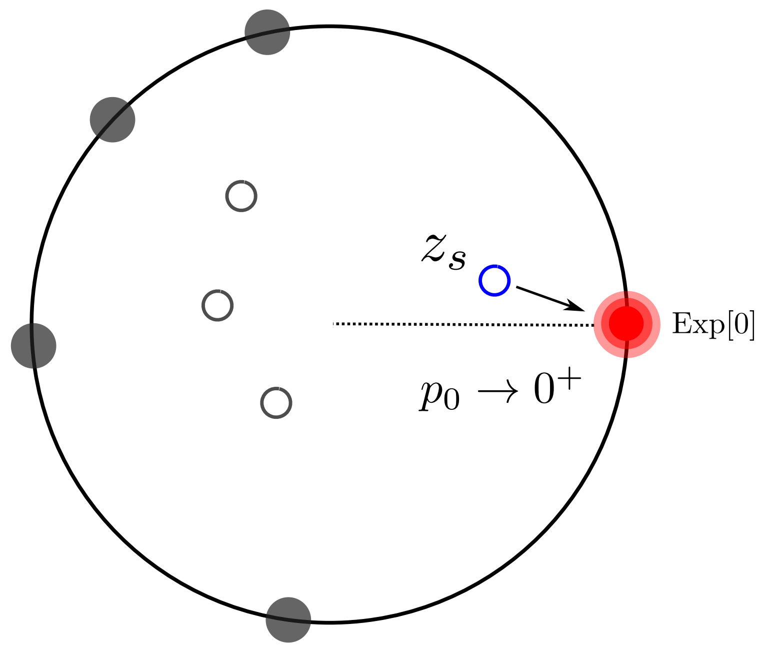
That is for the next talk.

Classical picture: Grunbaum et al. Charge Theory



Charge **magnitude:** $p_k = |\langle E_k|\psi_{in}|^2$. **Location:** $\exp(iE_k\tau)$.
Goal: Find the zeros z_i of the force field.
$\#$ of charges $= \langle n \rangle$.
Large fluctuations of n when $|z_i| \to 1$. **Only for return problem: a LOOP.**.
Algebra: $V(z) = \sum_k p_k \ln |e^{iE_k\tau} - z|$ and $\hat{\phi}(z) = z\Pi_{i=1}^{\omega-1} \frac{z-z_i}{z_i^* z-1}$.

Single Charge Theory

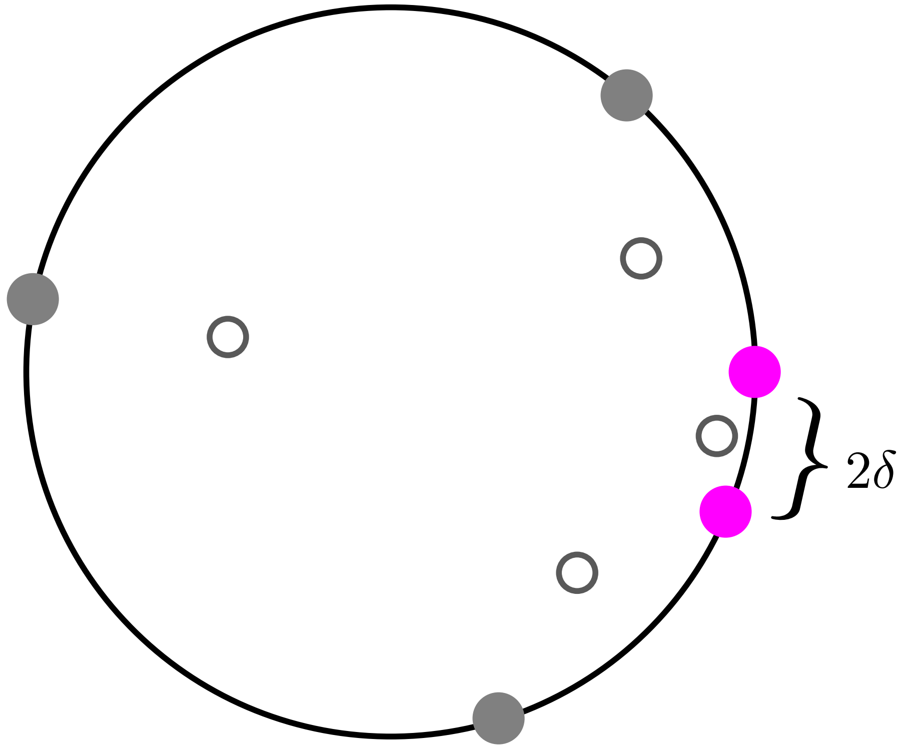


$$\boxed{z_s = 1 - \frac{p_0}{\sum_{j\neq 0} p_j/[1-\exp(iE_j\tau)]} \text{ and } \mathrm{Var}(n) \sim \frac{2|z_s|^2}{1-|z_s|^2}.}$$

Electrostatics: if one of the charges is weak we have a zero close to the unit circle.

QM: if $p_k = |\langle E_k|\psi_{in}|^2 << 1$ there is a component of the wave function that is difficult to detect, and this gives large quantum fluctuations.

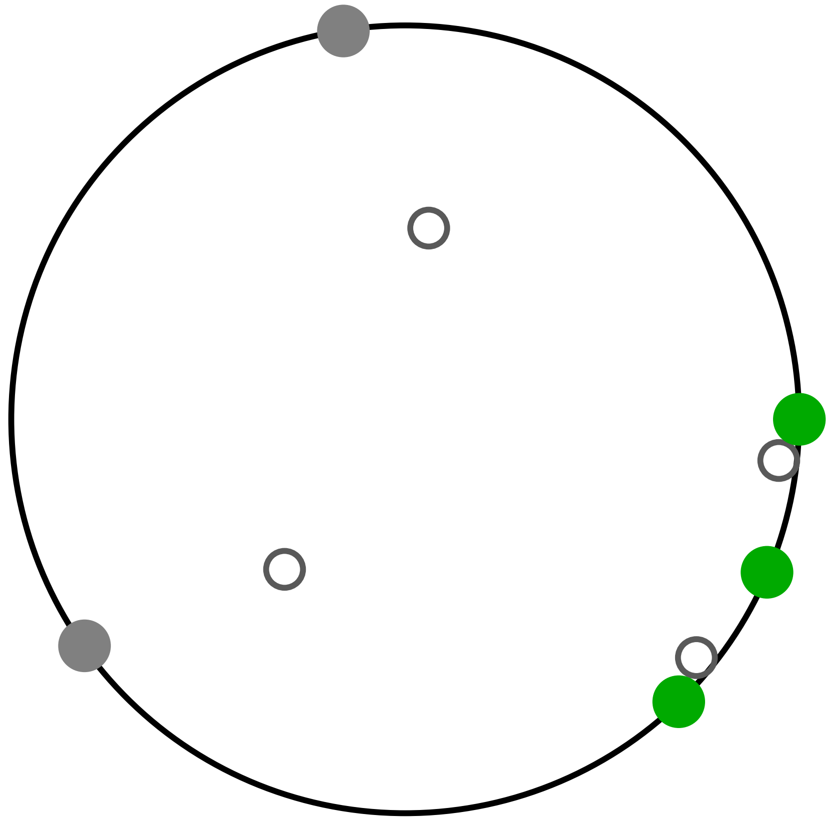
Two charge theory



$$\boxed{z_p \sim 1 + i\frac{p_1-p_2}{p_1+p_2}\delta + \left[\frac{4p_1p_2}{(p_1+p_2)^3}\sum_{j\neq 1,2}\frac{p_j}{\exp(iE_j\tau)-1} - \frac{1}{2}\right]\delta^2}$$

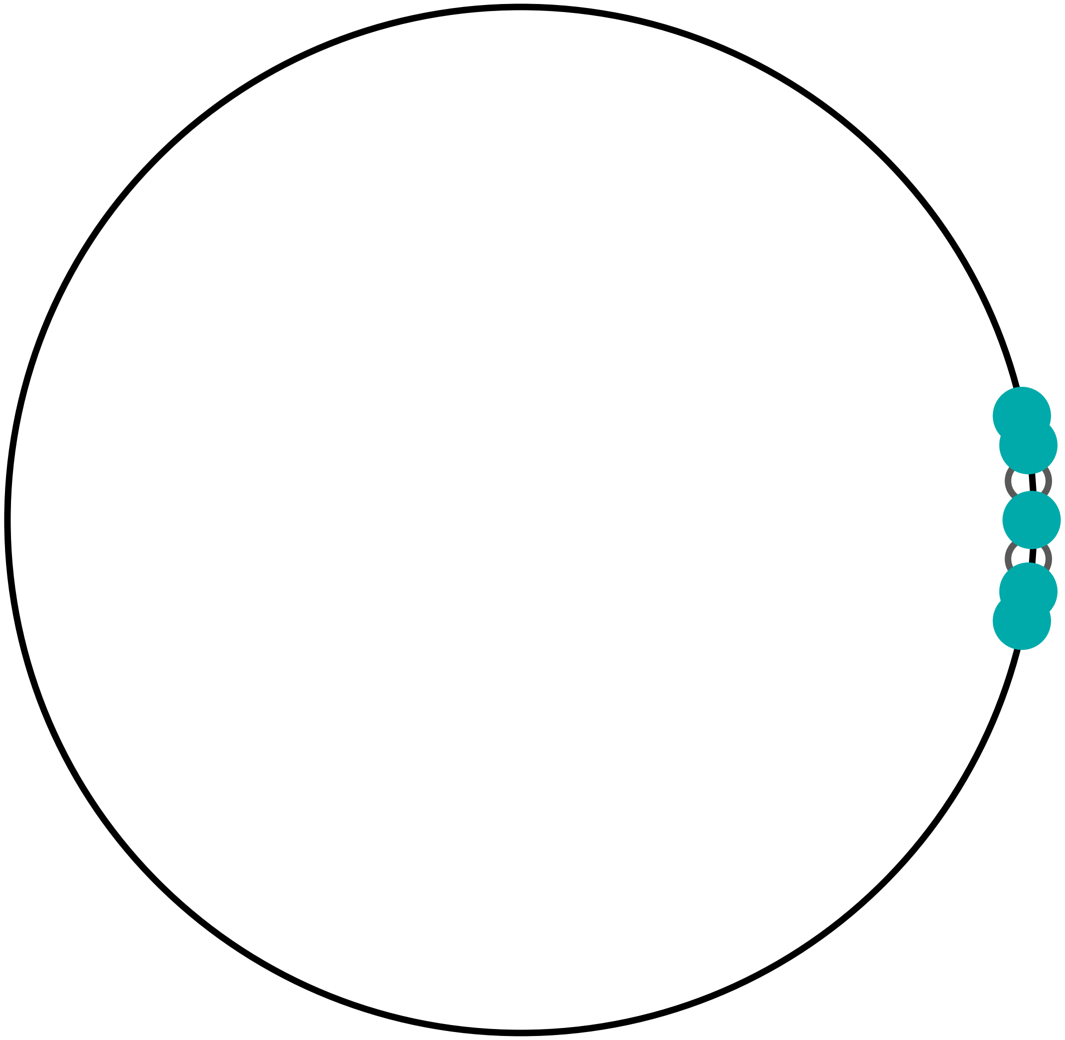
Var$(n) \sim \frac{2|z_p|^2}{1-|z_p|^2} = 2\frac{(p_1+p_2)^3}{p_1p_2\delta^2}$ is independent of the background.

Three charge theory



But why care? $\mathrm{Var}(n) \sim \sum_{\sigma=\pm} \frac{2|z_d^\sigma|^2}{1-|z_d^\sigma|^2} + \underbrace{\left(\frac{2z_d^+(z_d^-)^*}{1-z_d^+(z_d^-)^*} + c.c\right)}_{\text{Mixed}}$

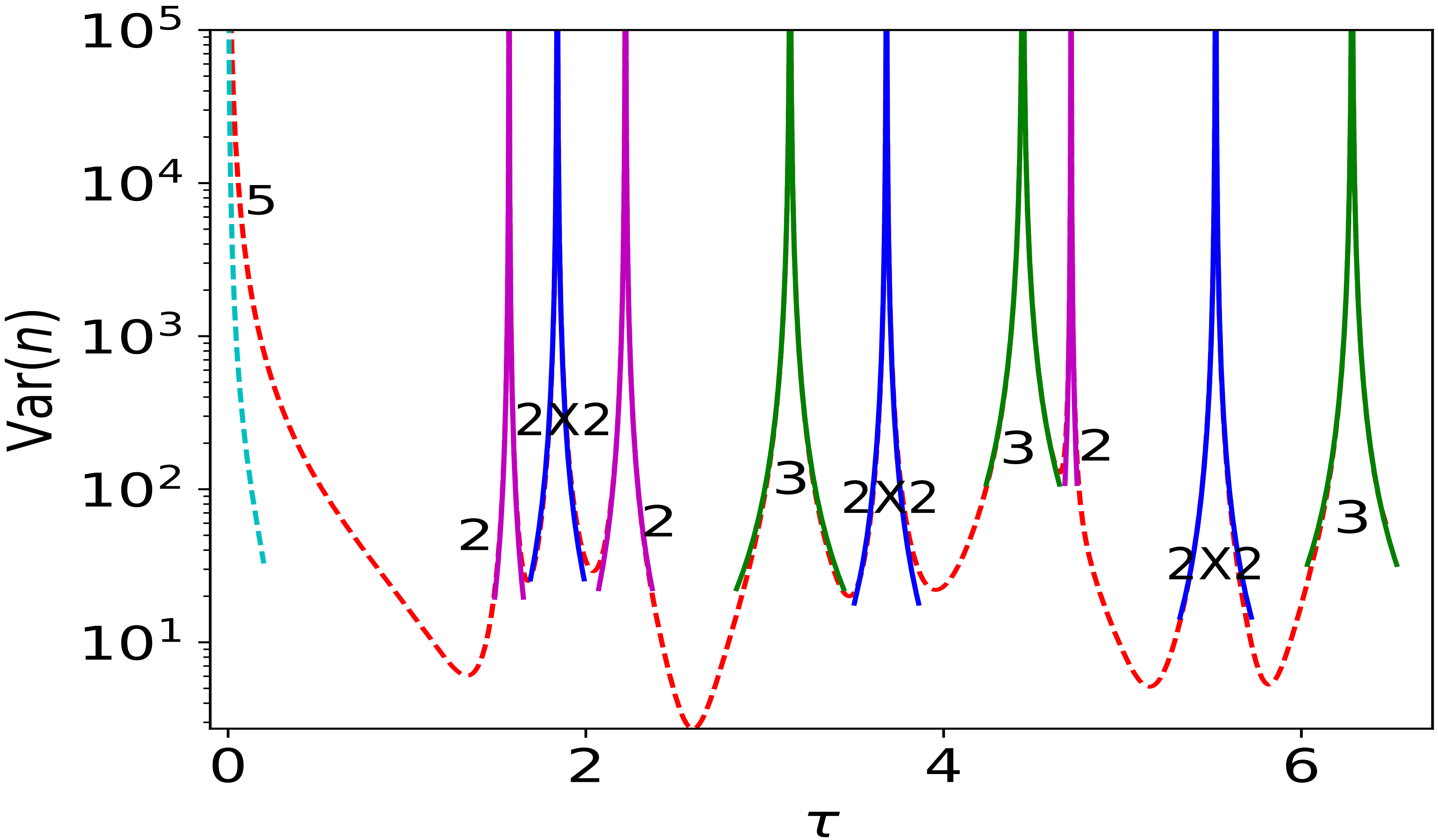
Charge configuration Zeno limit $\tau \to 0$



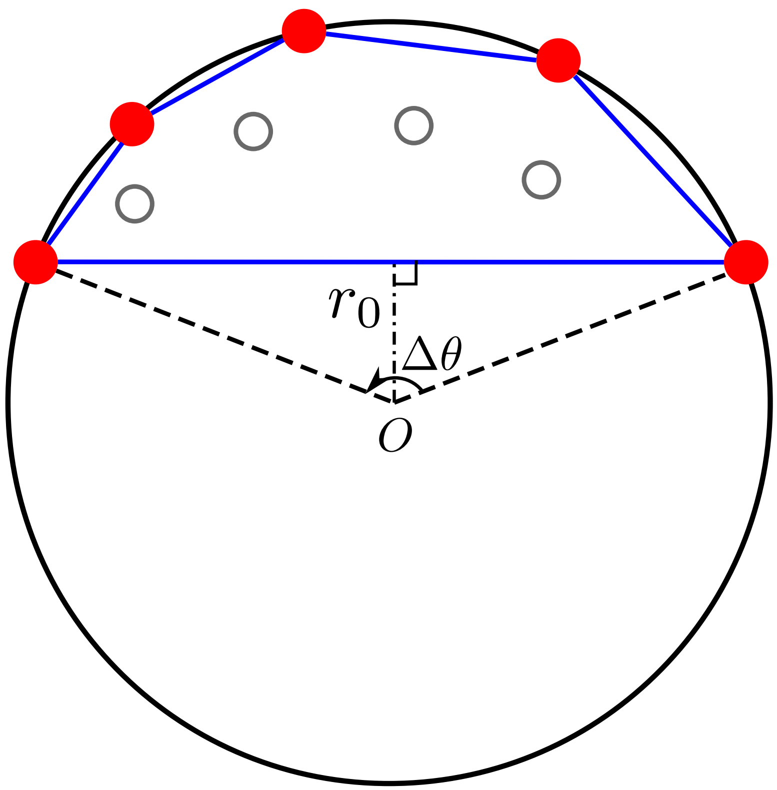
$\#$ of charges is number of distinct energy levels.

Ruoyu Yin, Ziegler, EB (in preparation)

Ring $L = 8$, $\langle n \rangle = 5$



Zeno Limit



Winding number for the return problem

- The generating function $\hat{\phi}(z) = \sum_{n=1}^{\infty} z^n \phi_n$.
- From the quantum renewal equation $\hat{\phi}(z) = \frac{\langle \psi_{in} | \hat{U}(z) | \psi_{in} \rangle}{1 + \langle \psi_{in} | \hat{U}(z) | \psi_{in} \rangle}$ with $\hat{U}(z) = \sum_{n=1}^{\infty} z^n \exp(-iH\tau n)$.
- Decompose in an energy basis: $|\psi_{in}\rangle = \sum_k \langle E_k | \psi_{in} \rangle | E_k \rangle$.
- Use $\exp(-iHn\tau)|E_k\rangle = \exp(-iE_k n\tau)|E_k\rangle$, sum a geometric series and use normalization $\sum_k p_k = 1$ here p_k is the probability of finding the initial condition in energy state k

$$\hat{\phi}(z) = \frac{\sum_k \frac{z p_k e^{-iE_k \tau}}{1 - z e^{-iE_k \tau}}}{\sum_k \frac{p_k}{1 - z e^{-iE_k \tau}}}.$$

- Rearrange, use $1/[\exp(ix) - 1] = [-1 - i\text{cot}(x/2)]/2$ and consider $z = \exp(i\theta)$, i.e. we are on the unit circle

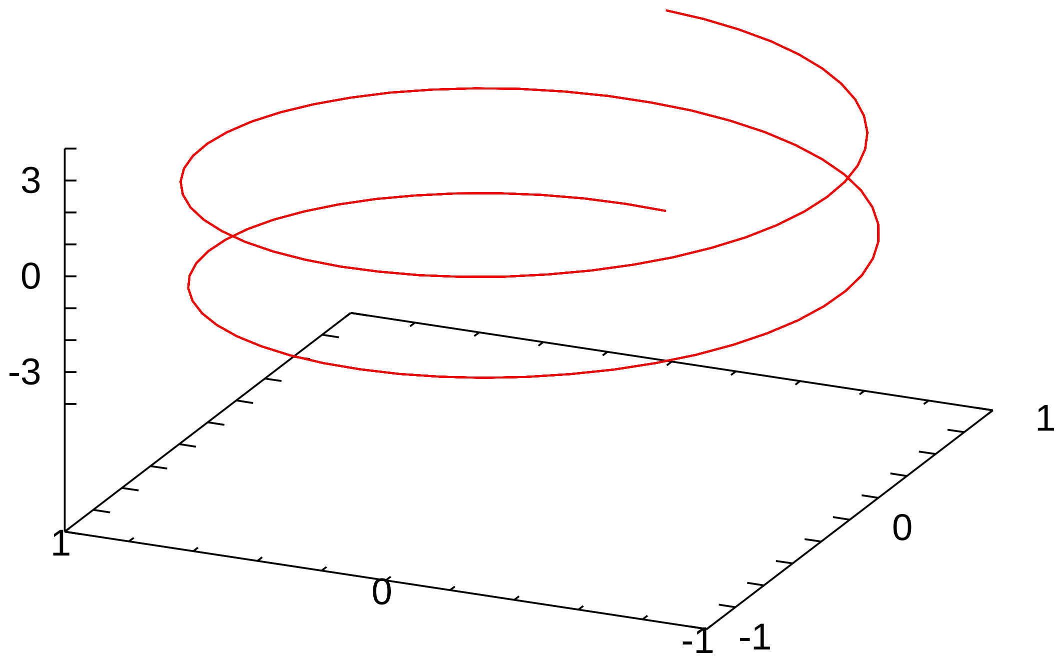
$$\hat{\phi}\left(e^{i\theta}\right) = \exp\left[i f(\theta)\right]$$

where

$$f(\theta) = 2\text{ArcTan}\left[\sum_k p_k \cot \frac{E_k \tau - \theta}{2}\right].$$

- From here: eventually the process is detected (for discrete spectrum, and return problem) and this follows from $|\exp[if(\theta)| = 1$.
- The average of n is an integer equal to the winding number.
- More specifically $\langle n \rangle$ is the number of distinct energy levels with $p_k \neq 0$ unless $\Delta E \tau = 2\pi j$.

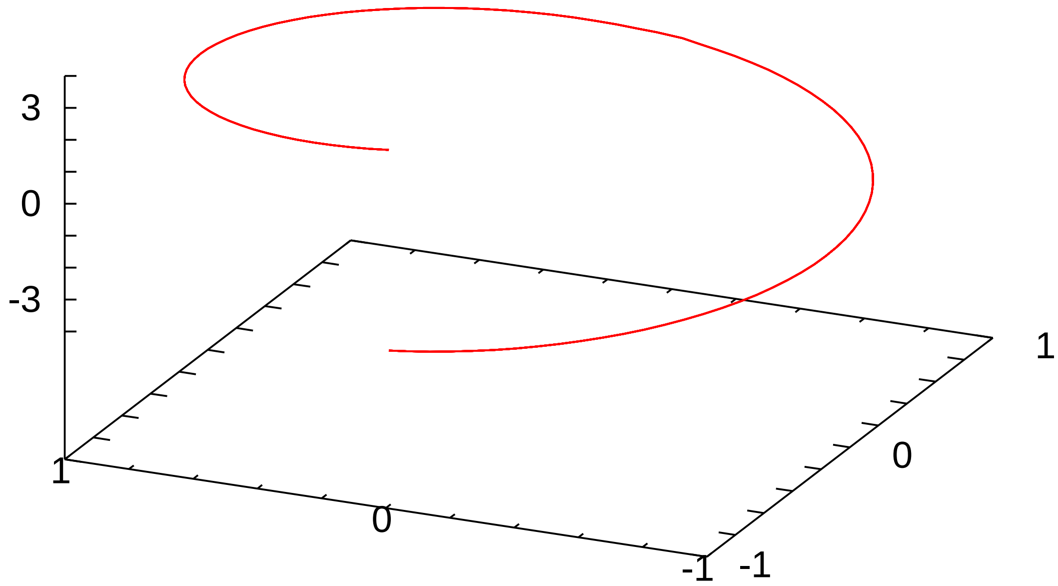
Winding up the generating function. RETURN problem.



Plot $\hat{\phi}(z)$ where $z = \exp(i\theta)$ and $-\pi < \theta < \pi$ Grünbaum et al. (2013).

Simple model of Y structure, number of distinct energy levels 3, but one energy state is orthogonal to the detected site, so effective dimension of the Hilbert space is 2. Here we consider the return to center of structure.

Topology modified at specific sampling times



When $\Delta E\tau = 2\pi$, winding number exhibits a discontinuity $\langle n\rangle = 2 \to 1$.

Generating function - basic formalism

$$\phi_n = \frac{1}{n!}\frac{d^n}{dz^n}\hat{\phi}(z)|_{z=0} \tag{1}$$

or

$$\phi_n = \frac{1}{2\pi i}\oint_C \hat{\phi}(z) z^{-n-1} dz \tag{2}$$

Probability of being detected:

$$1 - S_\infty = \sum_{n=1}^{\infty} F_n = \sum_{n=1}^{\infty} |\phi_n|^2 =$$

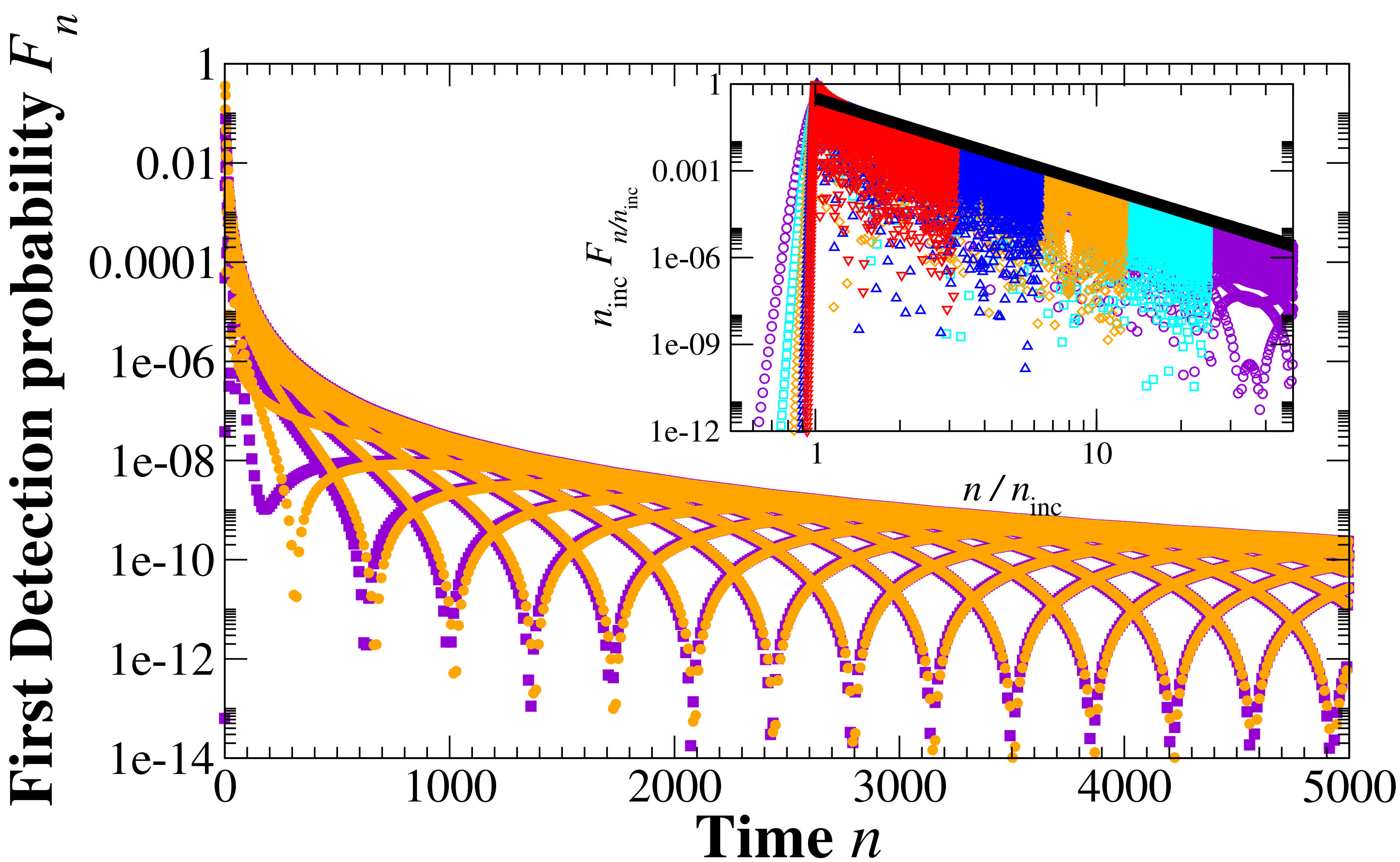
$$\frac{1}{2\pi}\int_0^{2\pi} \sum_{k=1}^{\infty} \phi_k e^{i\theta k} \sum_{l=1}^{\infty} \phi_l^* e^{-i\theta l} d\theta = \frac{1}{2\pi}\int_0^{2\pi} |\hat{\phi}(e^{i\theta})|^2 d\theta. \tag{3}$$

Similarly

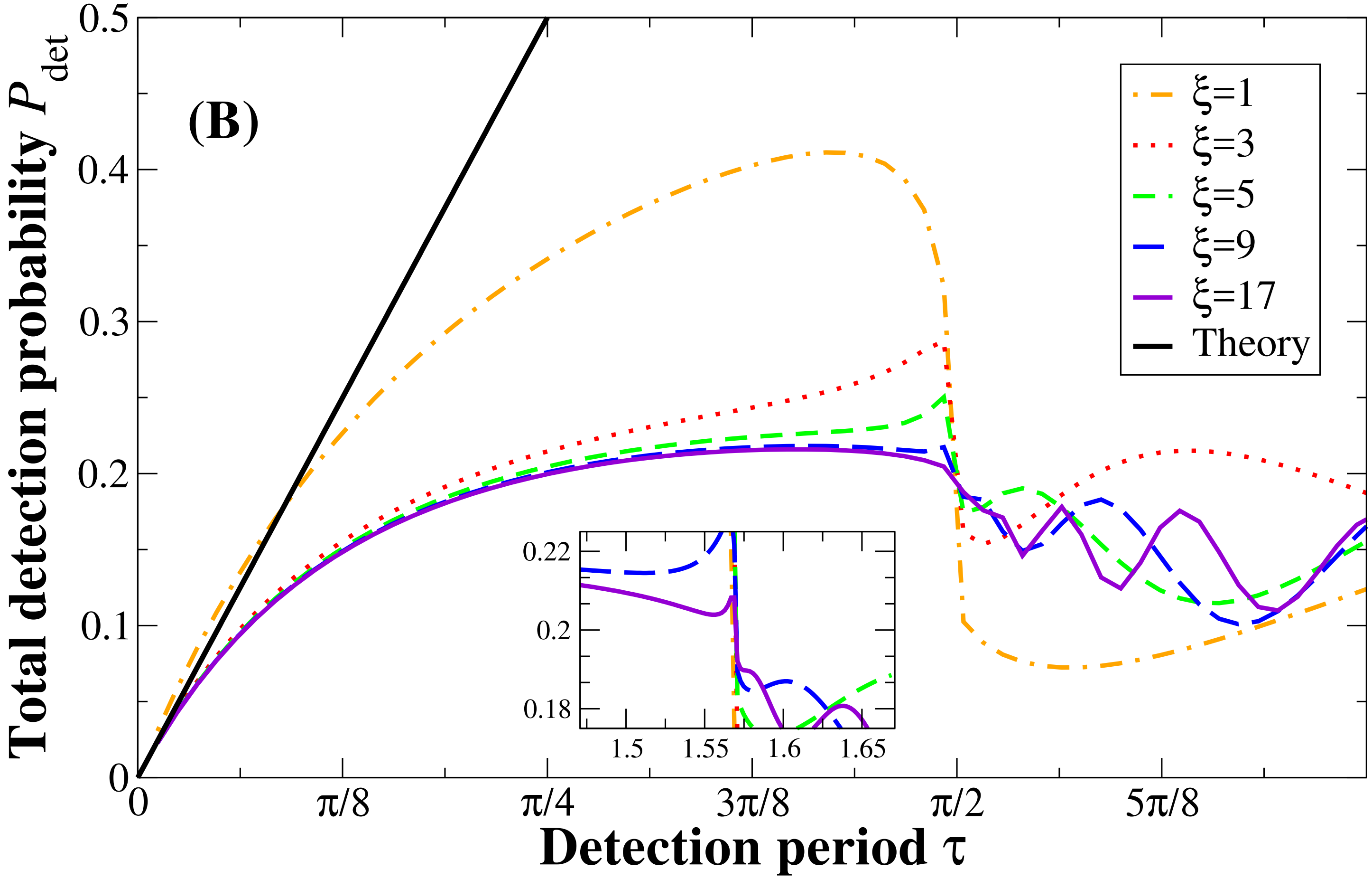
$$\langle n \rangle = \sum_{n=1}^{\infty} n F_n = \frac{1}{2\pi}\int_0^{2\pi} \left[\hat{\phi}\left(e^{i\theta}\right)\right]^* \left(-i\frac{\partial}{\partial\theta}\right) \hat{\phi}(e^{i\theta}) d\theta. \tag{4}$$

A shorthand notation is $\langle n\rangle = \langle\hat{\phi}| - i\partial_\theta|\hat{\phi}\rangle$.

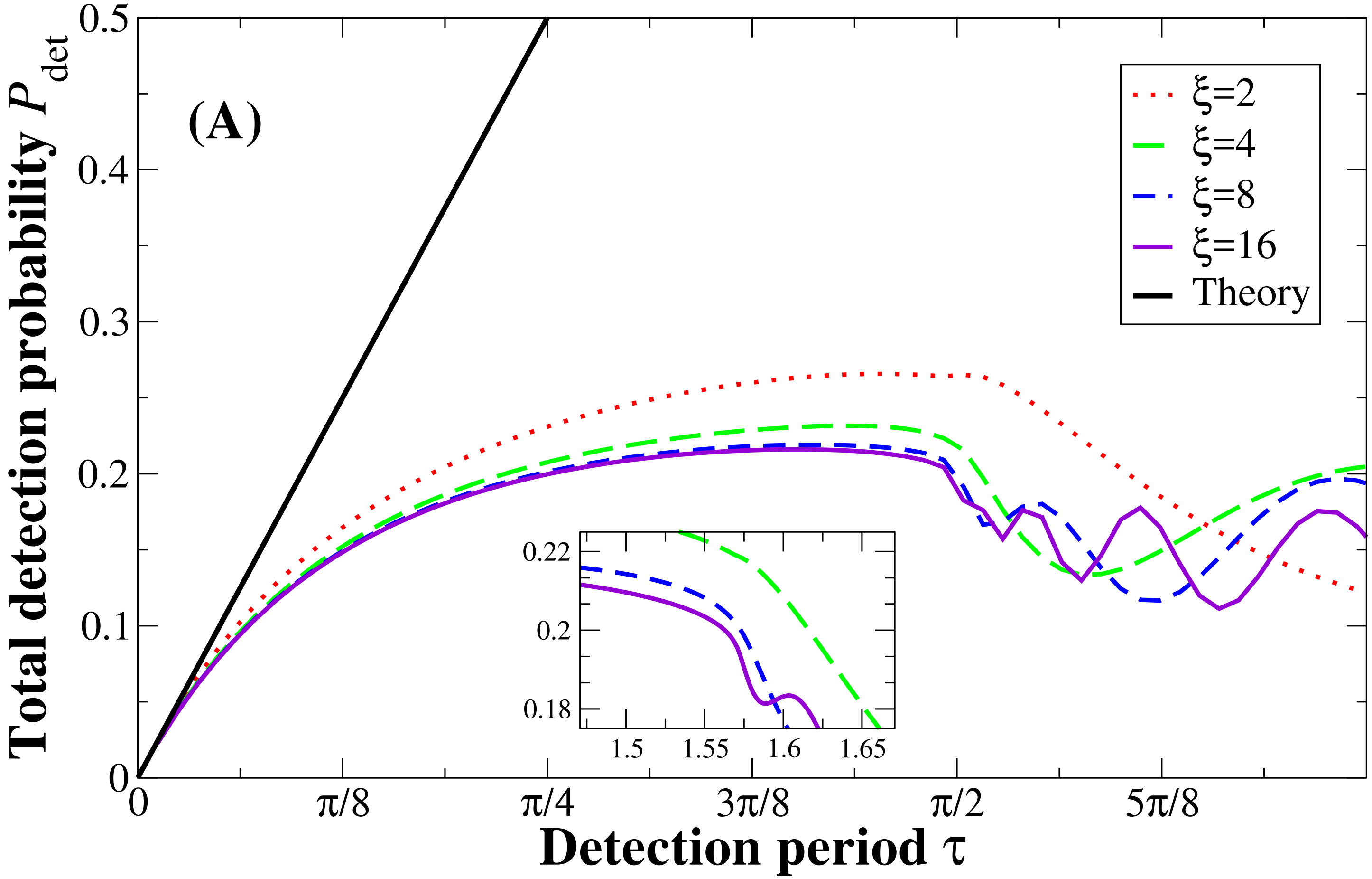
First Detection theory



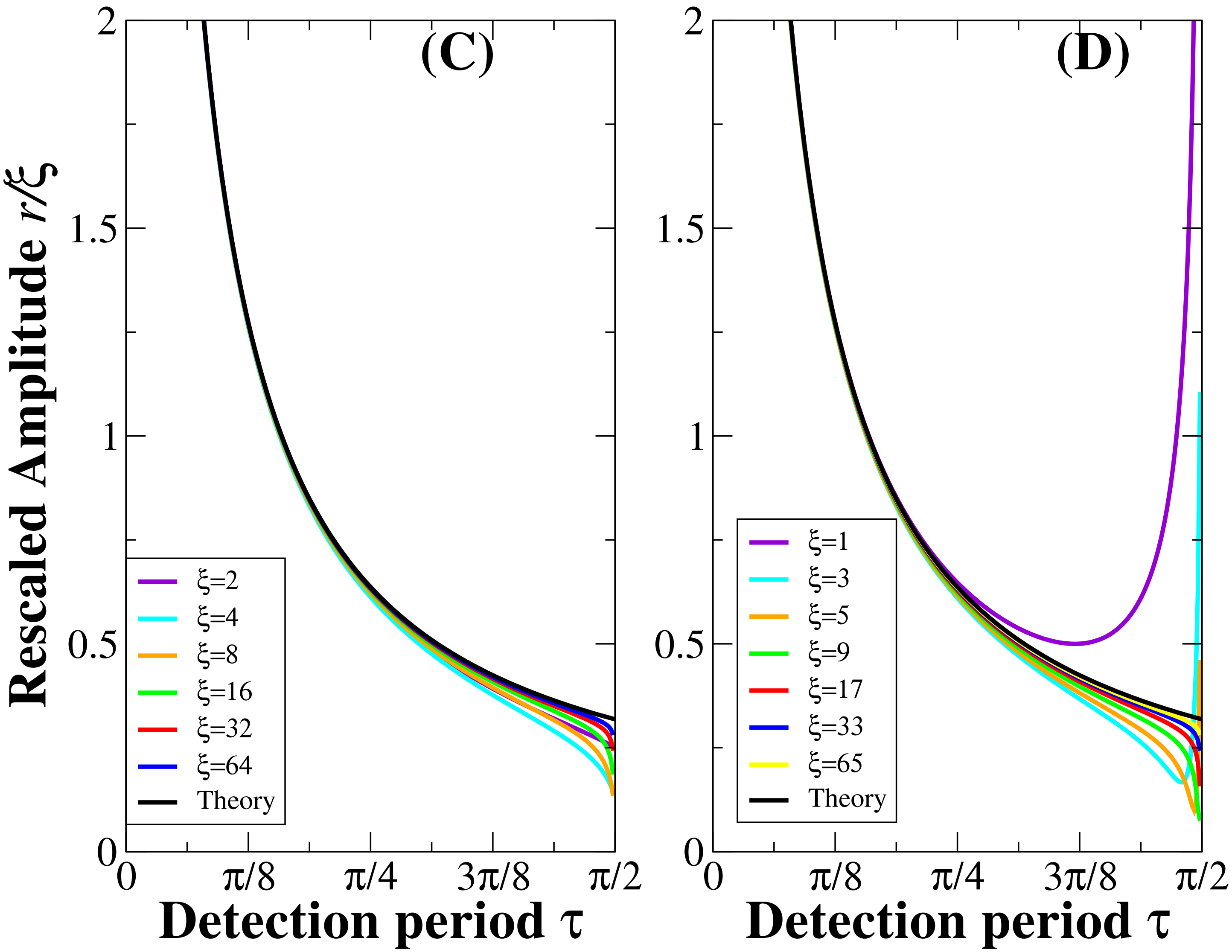
Total Detection



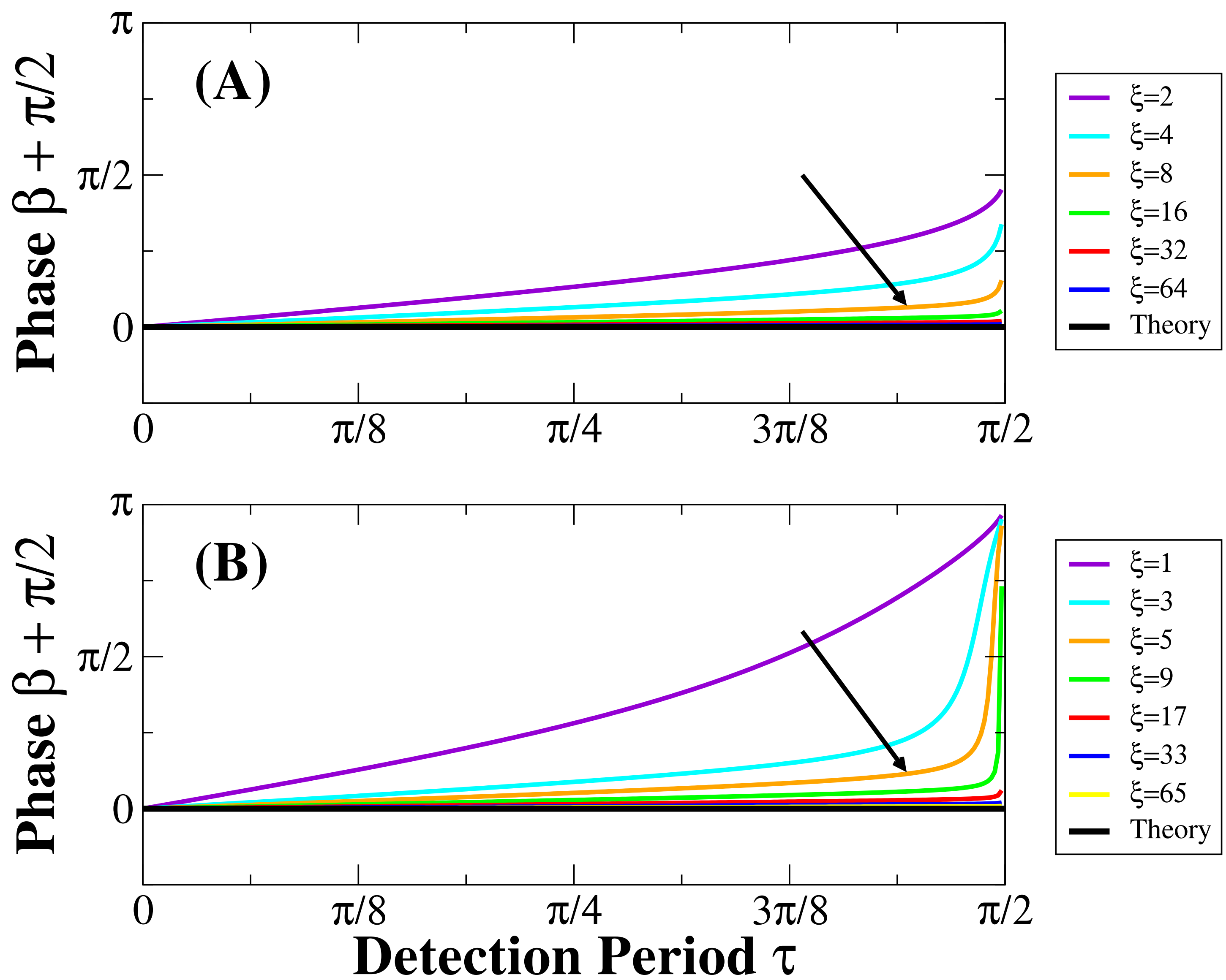
Total Detection



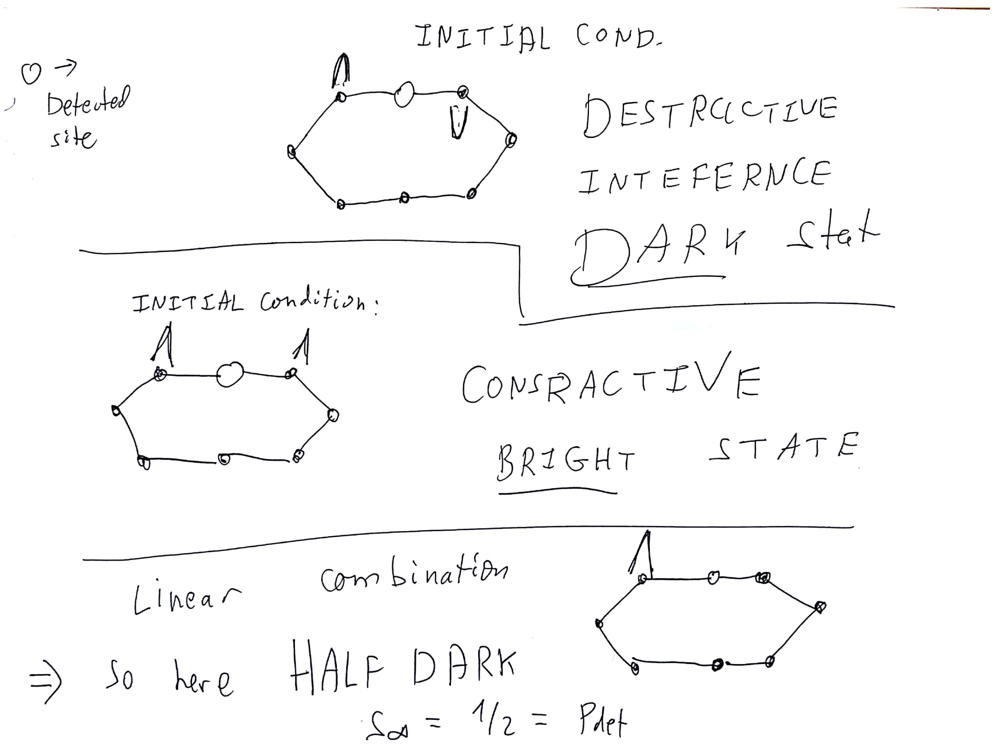
Prefactor Abs



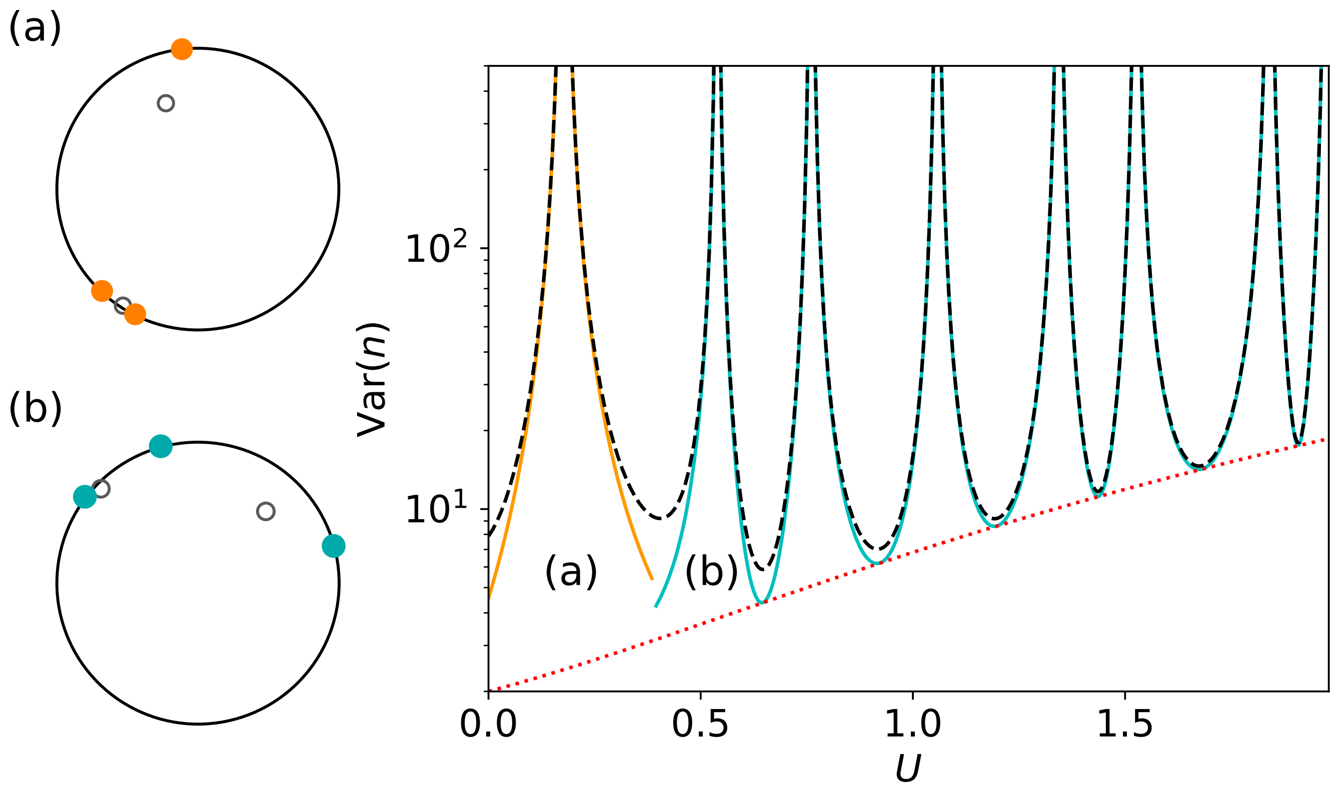
Prefactor Abs



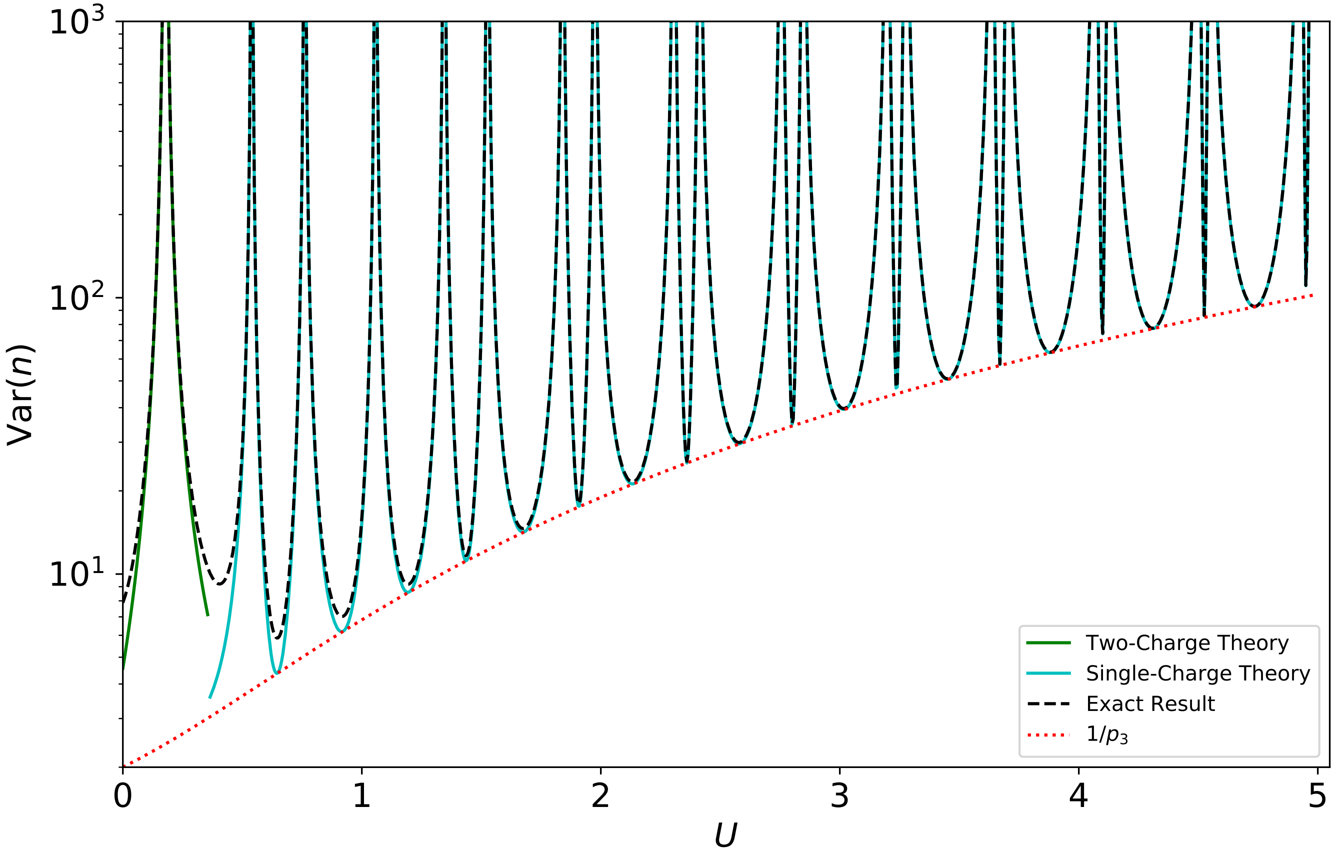
Why do we find dark states?

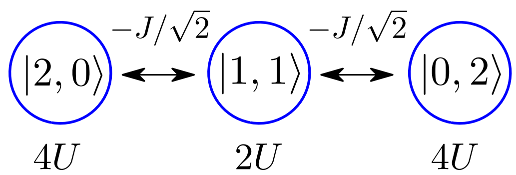


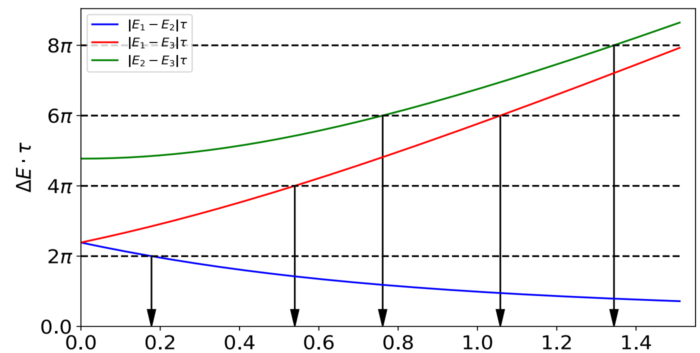
two boson model

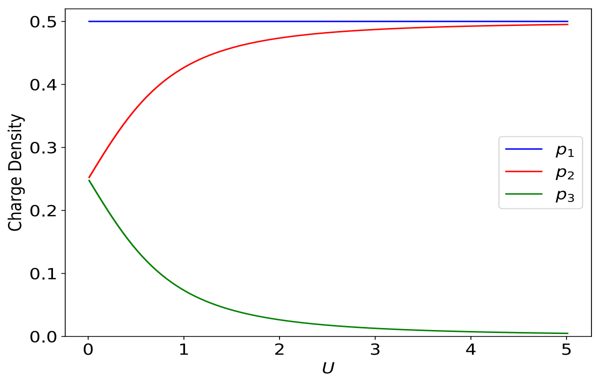


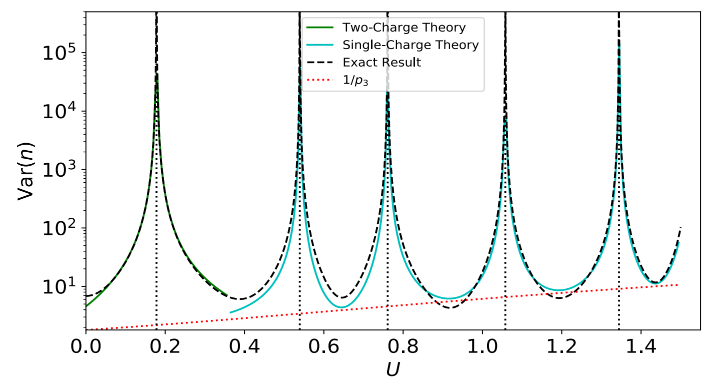
two boson model

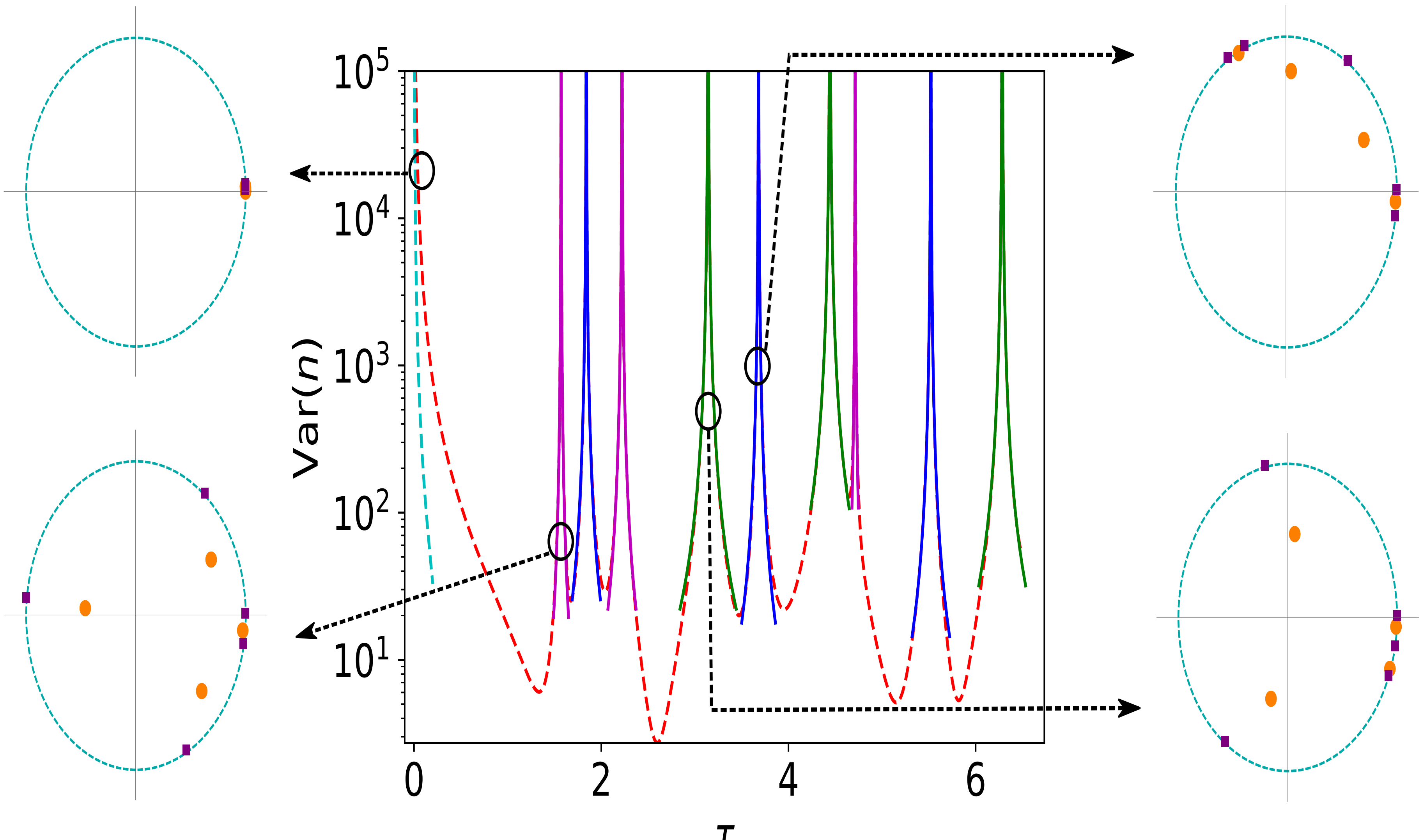



$-J/\sqrt{2}$
$-J/\sqrt{2}$
$|2,0\rangle$
$|1,1\rangle$
$|0,2\rangle$
$4U$
$2U$
$4U$


$|E_1-E_2|\tau$
$|E_1-E_3|\tau$
$|E_2-E_3|\tau$
$\Delta E\cdot\tau$
8π
6π
4π
2π
0.0
0.0 0.2 0.4 0.6 0.8 1.0 1.2 1.4

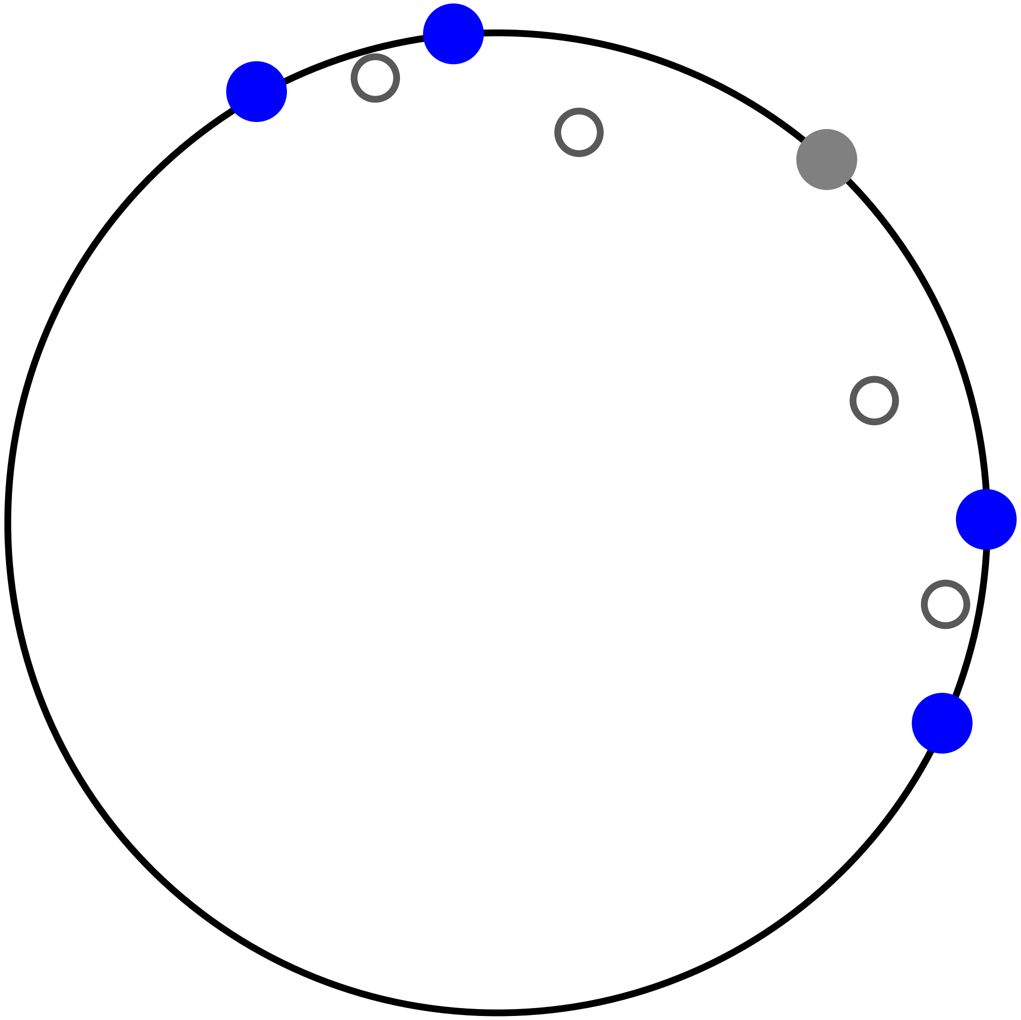

Charge Density
p_1
p_2
p_3
0.5 0.4 0.3 0.2 0.1 0.0
0 1 2 3 4 5
U


Two-Charge Theory
Single-Charge Theory
Exact Result
$1/p_3$
Var(n)
10^5 10^4 10^3 10^2 10^1
0.0 0.2 0.4 0.6 0.8 1.0 1.2 1.4
U

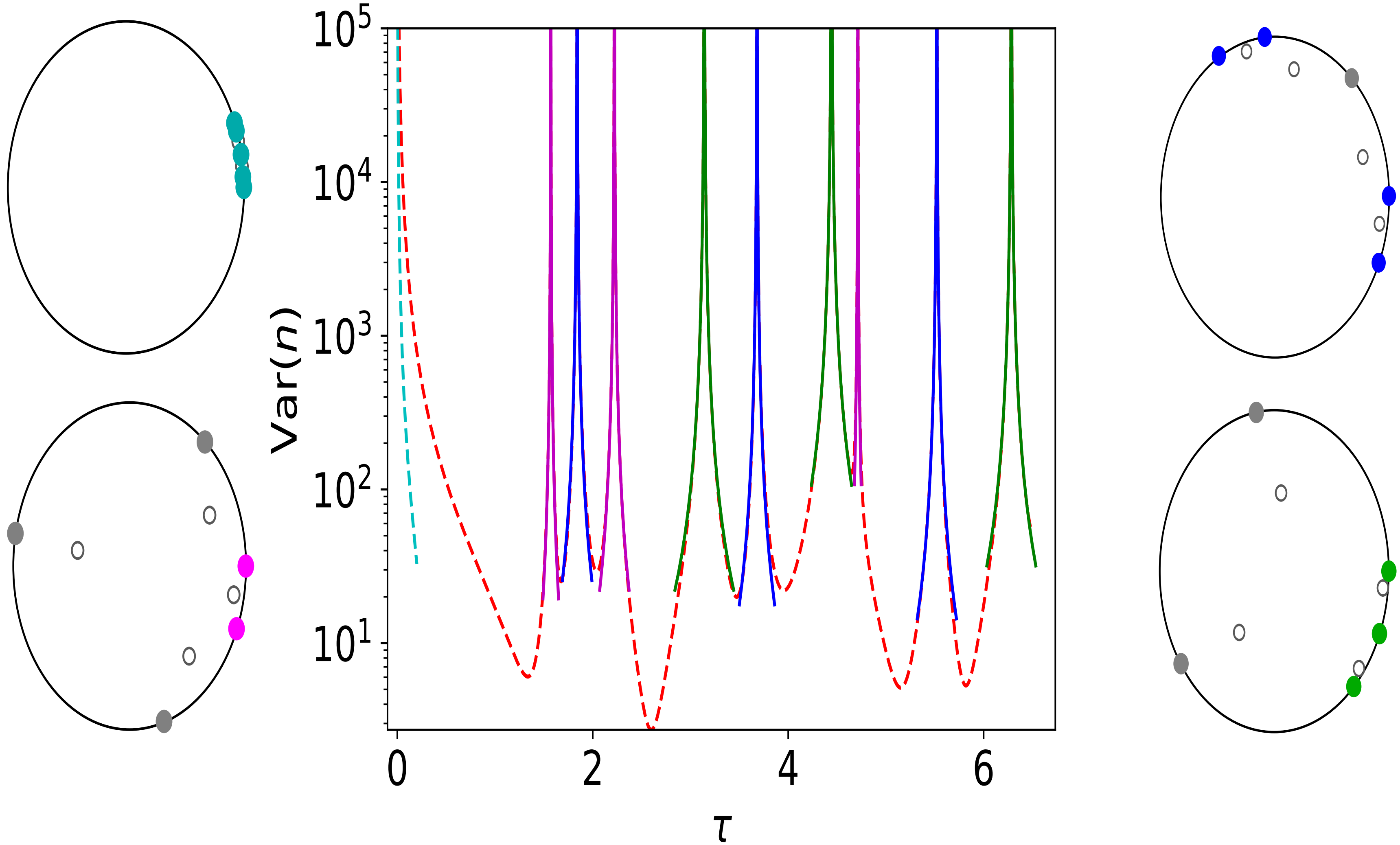


10^5
10^4
10^3
10^2
10^1
Var(n)
0
2
4
6
τ

Two charge theory



Ring



Connection to Classical Renewal Formula

Schrödinger in 1915 formulated the "renewal formula" for the first passage problem, relating the first passage probability $F(x,t)$ with the absorption-free probability distribution $P(x,t;x_0)$ for a random walker released at $x = x_0$ at time 0:

$$P(x,t;x_0) = \delta_{x,x_0}\delta_{t,0} + \sum_{t' \le t} F(x,t')P(x,t-t';x)$$

Taking a z-transform of this equation yields a solution for F in terms of the absorption-free propagator P:

$$P(x,z) = \delta_{x,x_0} + F(x,z)P(0,z) \Rightarrow F(x,z) = \begin{cases} \frac{P(x,z;x_0)}{P(x_0,z;x_0)}; & x \neq x_0 \\ 1 - \frac{1}{P(x_0,z;x_0)}; & x = x_0 \end{cases}$$

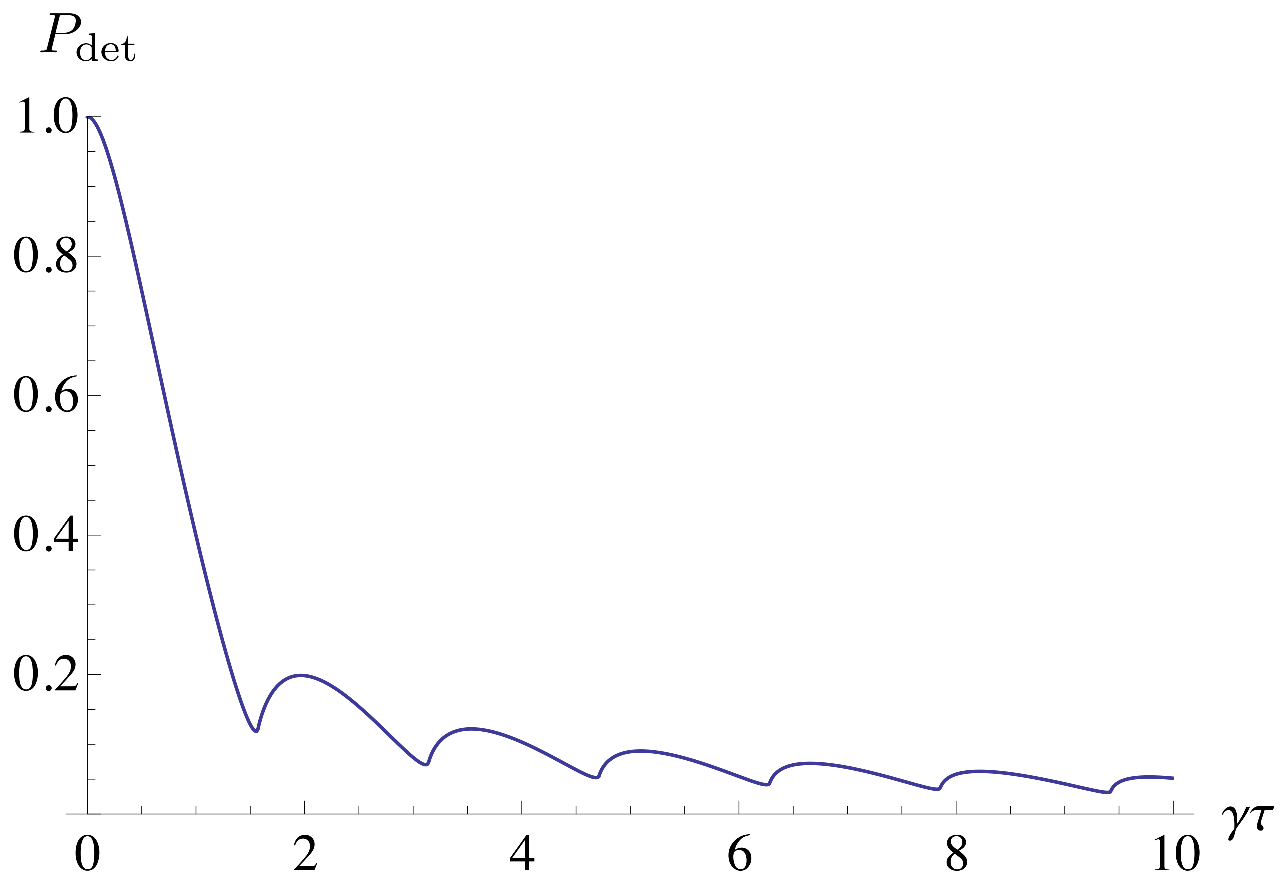
The parallel with our formula is clear

$$\hat{\phi}(z) = \Big\{ \langle x_M | \psi(z;x_0)\rangle / \langle x_M | \psi(z;x_M)\rangle; \;\; 1 - 1/\langle x_0 | \psi_f(z;x_0)\rangle \Big\}.$$

Marrying Schrödinger's Two Great Works!

Detection probability $P_{det} = \sum_{n>0} F_n$

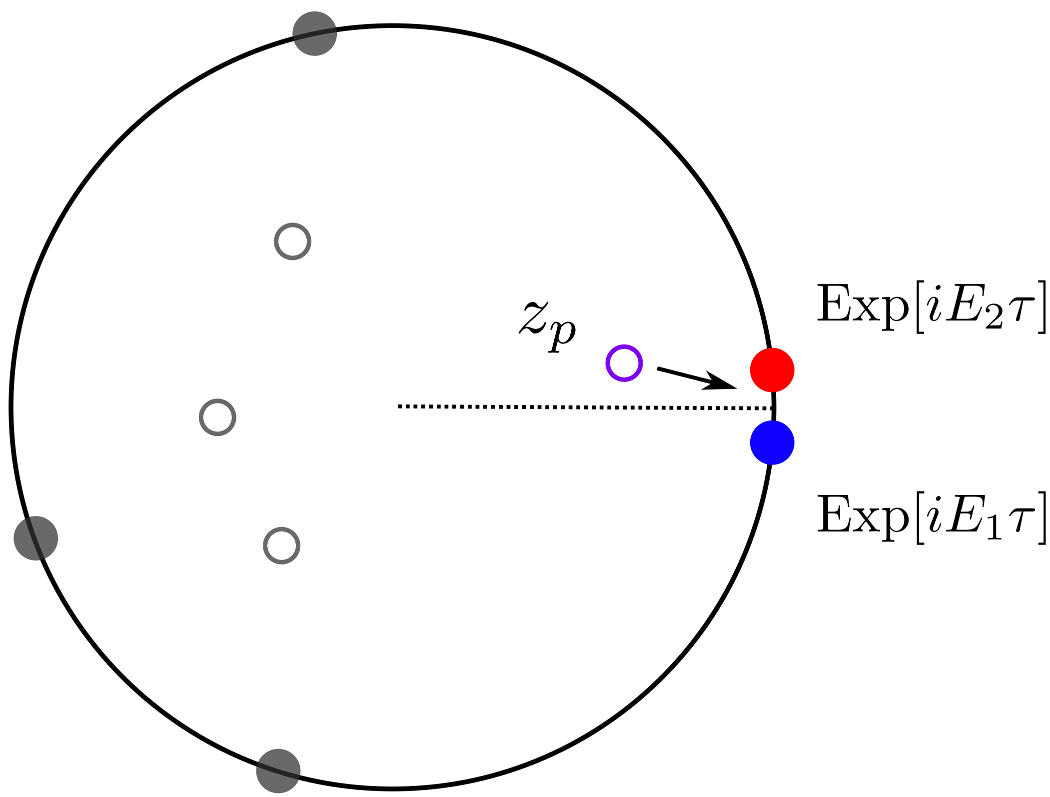
Unlike classical random walks in $1d$ probability of detecting the particle is not unity



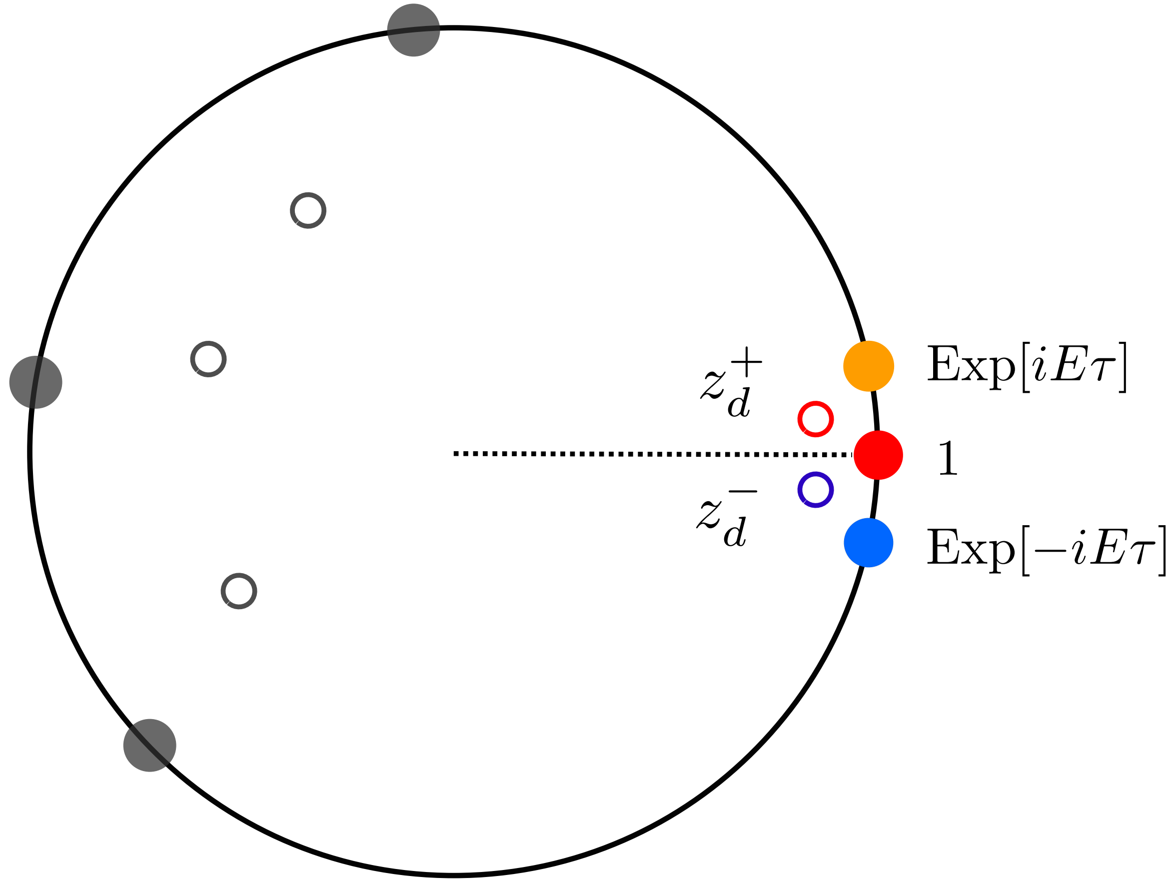
The Quantum Zeno Effect

- If $x_0 \neq x_M$ $\lim_{\tau \to 0} F_n = 0$. The particle is not detected.
- Since the measurement zeroes the wave function at x_M, if we measure too often, the particle will never get to x_M.
- "A watched quantum pot never boils"
- Mathematically it is easy to show:

$$\lim_{\tau \to 0} \hat{\phi}(z) = z \langle X_M | \psi_{in} \rangle .$$



z_p
$\mathrm{Exp}[iE_2\tau]$
$\mathrm{Exp}[iE_1\tau]$


z_d^+
$\mathrm{Exp}[iE\tau]$
1
z_d^-
$\mathrm{Exp}[-iE\tau]$