Dynamical mean field theory of neural networks with power-law disorder

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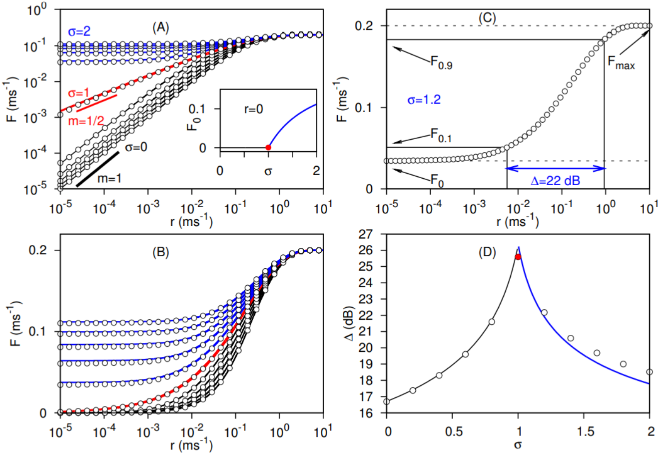
Background

Computation at the edge of chaos: optimal dynamical range



Kinouchi O. et al. Nat. Phys. 2, 348351 (2006)

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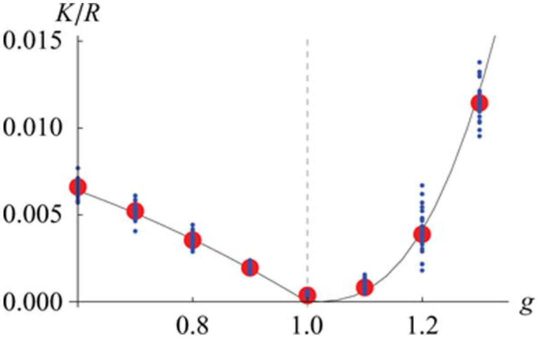
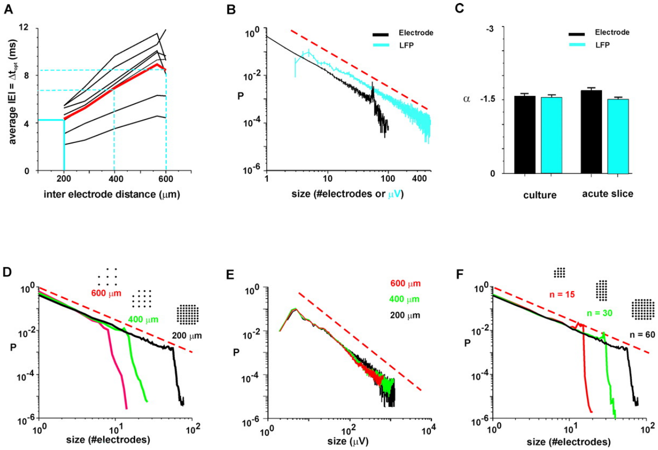


Toyoizumi T. et al. Phys. Rev. E 84, 051908 (2011).

Background

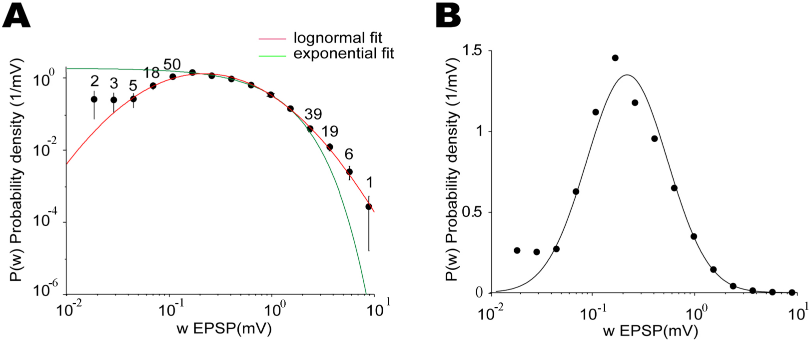
Critical brain hypothesis: neuronal avalanches



Beggs J.M. et al. J. Neurosci. 23 (35) 11167-11177 (2003).

Background

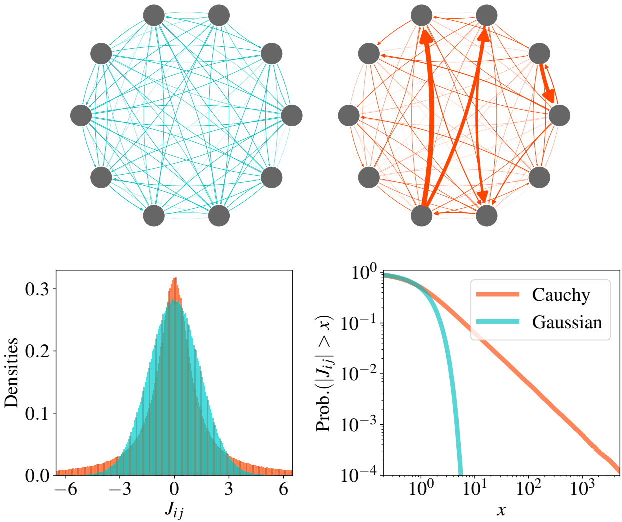
Distribution of synaptic connection strengths (EPSP amplitude)



Song S. et al. PLoS Biol. 3(3): e68 (2005).

Model

Connectivity models



Model

Connectivity models

Cauchy model

$$\rho(J_{ij}) = \frac{1}{\pi} \frac{g/N}{(g/N)^2 + J_{ij}^2} \quad (1)$$

Fully connected Gaussian model

$$J_{ij} \sim \mathcal{N}(0, g^2/N) \quad (2)$$

Sparse Gaussian model with K incoming connections per neuron

$$J_{ij} \sim \mathcal{N}(0, g^2/K) \quad (3)$$

Model

Discrete-time random recurrent neural network

$$x_i(t+1) = \sum_{j=1}^{N} J_{ij}\phi(x_j(t)) \quad (4)$$

- Symmetric **J**: relaxation of a global energy function, corresponds to the spin-glass Hamiltonian.
- Asymmetric **J** (uncorrelated J_{ij} and J_{ji}): can be chaotic.

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Simple results

Linear stability analysis

Let $\phi(x) \approx ax$ around $x = 0$

- Gaussian network chaotic (or unstable) for $ag > 1$,

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Let $\phi(x) \approx ax$ around $x = 0$

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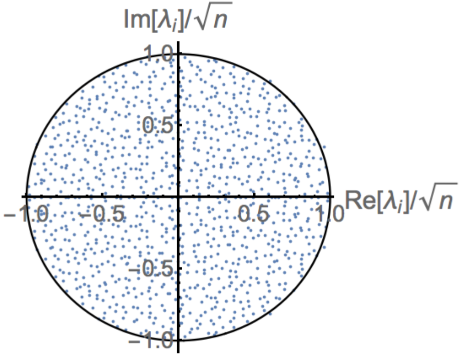
- Cauchy network always chaotic (or unstable),
- Related to the distributions of eigenvalues



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Linear stability analysis: problem with thresholds

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- Most neurons inactive in the absence of inputs

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Linear stability analysis: problem with thresholds

- Thresholds
- Most neurons inactive in the absence of inputs
- Here

$$\phi(x) = \begin{cases} 1, & \text{for } x > \theta \\ 0, & \text{for } x \leq \theta \end{cases} \tag{5}$$

Mean field

Sketch of the derivation

- Order parameter: the average network activty

$$m(t) = \frac{1}{N} \sum_{i=1}^{N} |\phi(x_i(t))| \tag{6}$$

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- J_{ij} and $x_i(t)$ described by the stable distributions with the characteristic function

$$\Phi_J(k) = e^{-\gamma |k|} \tag{7}$$

Mean field

Results

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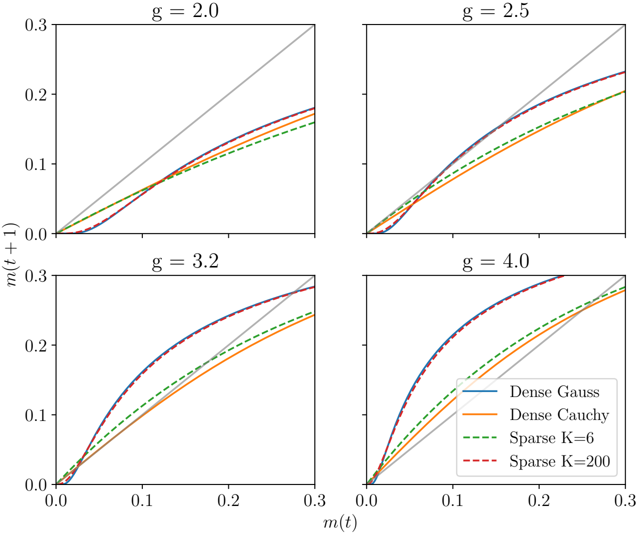
- Sparse Gaussian model

$$m(t+1) = \frac{1}{2}\sum_{n=1}^{K}\binom{K}{n} m(t)^n (1-m(t))^{K-n}\left(1 - \operatorname{erf}\left(\frac{\theta\sqrt{K}}{\sqrt{2ng}}\right)\right) \tag{10}$$

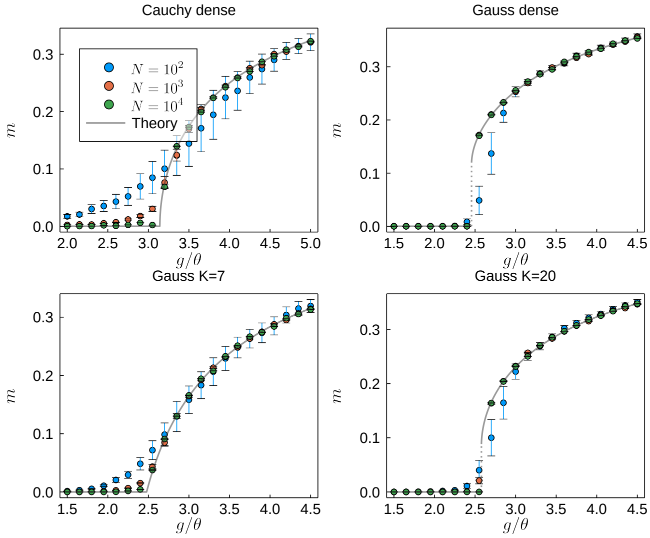
Mean field

Theoretical predictions



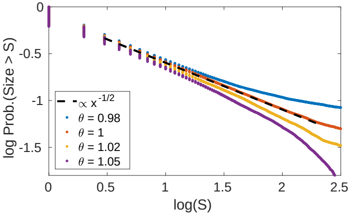
Mean field

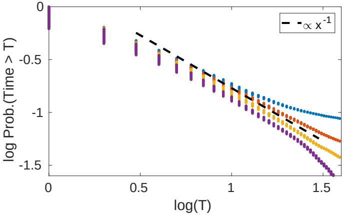
Theoretical predictions vs. simulations



Critical behavior

Scale-free avalanches & mapping to the branching process





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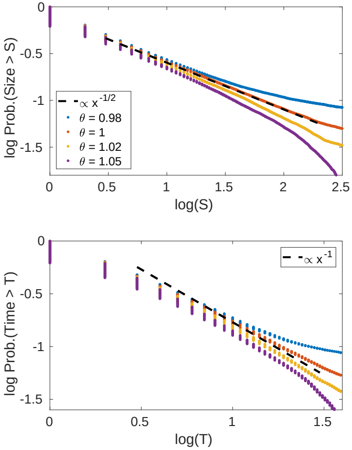

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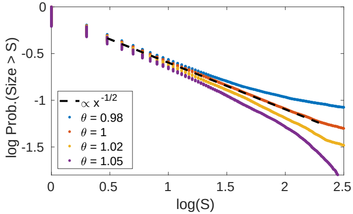

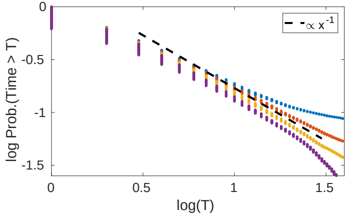

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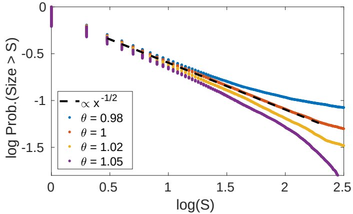

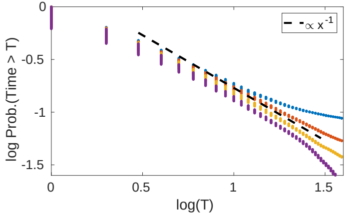

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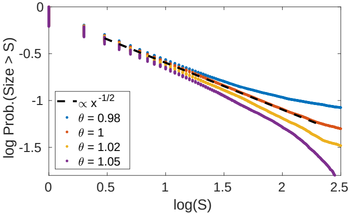

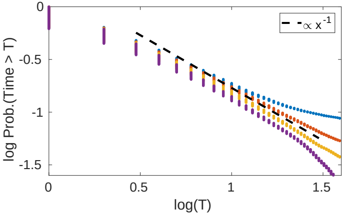

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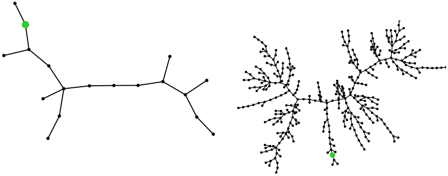
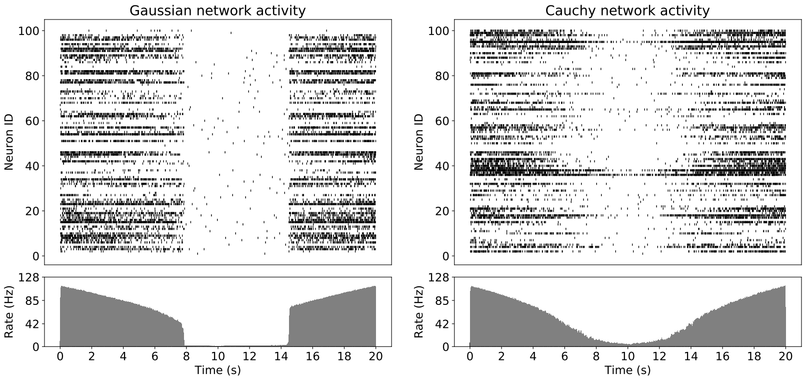


Spiking neurons

Simulations of 10^4 *leaky integrate-and-fire* neurons.



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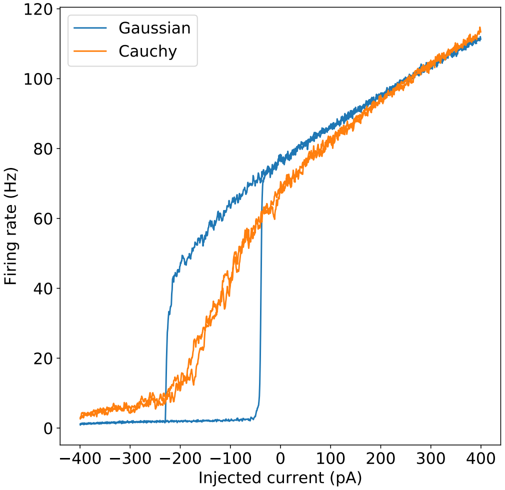

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- Novel model of connectivity with heavy tails: Cauchy network
- Cauchy network: continuous transition to chaos, low activity levels, scale-free avalanches
- Strong synapses as a backbone, weak synapses as a pool of potential connections (weakly informative sparsity prior)