Of Brains and Markets

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Outline

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Of Markets . . .

- Stylized Facts of Market Dynamics
  - Fat tailed (leptocurtic) return distributions
  - Fast decorrelation of asset returns
    Slow decorrelation of absolute returns
  - Long range correlations of volatility
    (volatility clustering).

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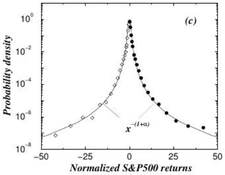

S&P 500 return distributions

(Gopikrishnan et al PRE, 1999)

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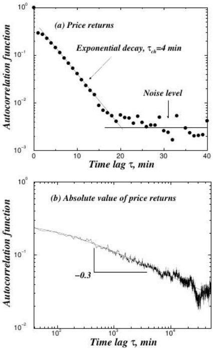

Auto-correlations of returns
and absolute returns

(Gopikrishnan et al PRE, 1999)

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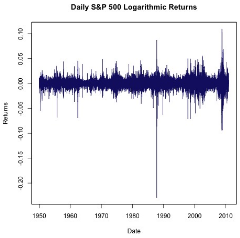


Daily returns S&P 500

Geometric Brownian Motion

- Geometric Brownian motion model (GBM)

$$\mathrm{d}S_i(t) = S_i(t)\left[\mu_i \mathrm{d}t + \sigma_i \mathrm{d}W_i(t)\right]$$

  exhibiting

  - log-prices follow diffusive motion with drift
  - Gaussian log-return distributions
  - no correlations of volatility

- Is the "harmonic oscillator" of Financial Mathematics.
- Is at the heart of the Black-Scholes option pricing method.
- Does not reproduce the key empirical facts of market dynamics.
- Yet, with modifications still widely used in financial industry.

Fixes

- Phenomenological
  - Replace Brownian (Gaussian) increments in GBM by fat tailed increments (e.g. Lévy: Mantegna and Stanley, 1994)
  - Add evolution of volatilities ⇒ ARCH/GARCH/stoch. volatility (Engle, 1982; Engle and Bollerslev, 1986; Heston, 1993)
  - ...

    Typically single asset descriptions; no systemic perspective.

- Agent based models
  - e.g. Minority Game (Challet and Zhang, 1997)
  - Percolation models (Stauffer et al 1998, Cont and Bouchaud 2000)
  - Ising models of interacting agents (Iori, 1999; Da Silva Stauffer 2001)
  - ...

    All need fine-tuning of parameters to reproduce stylized facts.

- **Somehow unsatisfactory.**

Stepping Back – A Gedanken-Experiment

- **Question**
  Can we, just by looking at the basic structure of the problem of describing market dynamics, obtain guidance about fundamental properties **any** good model of market dynamics should have?

- To answer this question, let us perform a Gedanken-Experiment. It runs like this:

- Suppose I knew everything about markets, and when I say this, I mean **really everything**!

Stepping Back – A Gedanken-Experiment

- I would write down the complete set of dynamical equations describing all processes governing a market.

  (basic econmic laws, influence of supply and demand, effect of regulatory frameworks, psychology of traders, financial positions of trading agents, laws of order book dynamics, ...).

- Suppose that I would integrate out all degrees of freedom from my equations, except prices of assets traded in the market.

- Which properties would the reduced model necessarily have?

- It would
  - exhibit interactions between prices
  - exhibit a non-Markovian dynamics

- $\Rightarrow$: Formulate the simplest model with these properties.

RK & P Neu, J Phys A (2008); K. Anand, J Khedair & RK Phys Rev E (2018)

A Minimal Model of Interacting Prices – iGBM

- Generalization

$$\begin{aligned} \mathrm{d}u_i(t) &= I_i \mathrm{d}t + \sigma_i \mathrm{d}W_i(t) \\ &\quad + \Big[-\kappa_i u_i(t) + \sum_j J_{ij}\, \overline{g}_j(t) + \sigma_0 u_0(t) \Big] \mathrm{d}t \ , \\ \overline{g}_j(t) &= \int^t M(t-s)\, g(u_j(s)) \end{aligned}$$

$\Leftrightarrow$ interacting geometric Brownian motion model (iGBM),

with

- the κ_i producing a mean reversion effect,
- the J_{ij} describing strengths of interactions between assets,
- the $g = g(\cdot)$ denoting non-linear functions (e.g. sigmoid) describing the nature of the feedback,
- the $I_i = \mu_i - \frac{1}{2}\sigma_i^2$ (Ito)
- the $u_0(t)$ assumed to be a slow process describing the evolution of macro-economic conditions (model as slow OU process)

iGBM and Neural Networks — Brains and Markets

- iGBM and Neural Netowrks

$$\begin{aligned} \mathrm{d}u_i(t) &= I_i\mathrm{d}t + \sigma_i \mathrm{d}W_i(t) \\ &\quad + \Big[-\kappa_i u_i(t) + \sum_j J_{ij}\,\overline{g}_j(t) + \sigma_0 u_0(t) \Big]\mathrm{d}t\ . \\ \overline{g}_j(t) &= \int^t M(t-s)\,g(u_j(s)) \end{aligned}$$

- Describes dynamics of a network of graded response neurons, with
  - the u_i denoting trans-membrane voltages,
  - the κ_i describing leakage across the membrane,
  - the J_{ij} denoting synaptic couplings,
  - the $g(\cdot)$ being neural response functions
  - the I_i describing external signals.
  - the function $u_0(t)$ representing the effect of neuro-modulators.

Outline

Analysis (Synthetic Random System)

- Use generating functionals $(n_i(t) = g(u_i(t)))$

$$Z[\ell|u_0] = \left\langle \exp\left\{ -\mathrm{i} \int \mathrm{d}t \sum_i \ell_i(t) n_i(t) \right\} \right\rangle ,$$

- Averaging over couplings maps problem onto a family of effective single node problems,

$$\begin{aligned} \dot{u}_\vartheta(t) &= -\kappa\, u_\vartheta(t) + I + J_0\, m(t) + \sigma_0 u_0(t) \\ &\quad + \alpha\, J^2 \int_0^t \mathrm{d}s\, G(t,\, s)\, n_\vartheta(s) + \phi_\vartheta(t) , \end{aligned}$$

with $\vartheta \equiv (I, \kappa, \sigma)$ used as shorthand for site-random quantities. Here ϕ_ϑ is couloured noise with

$$\begin{aligned} \langle\, \phi_\vartheta(t) \,\rangle &= 0 \\ \langle\, \phi_\vartheta(t)\, \phi_{\vartheta'}(s) \,\rangle &= \delta_{\vartheta,\vartheta'} \Big[\sigma^2\, \delta(t - s) + J^2 q(t,s) \Big] . \end{aligned}$$

Order-parameters are coupled through a set of self-consistency equations.

Self-Consistency Equations

- Self-consistency equations, $(n_\vartheta(t) = g(u_\vartheta(t))$

$$
\begin{aligned}
m(t) &= \left\langle \langle n_\vartheta(t) \rangle_{\phi_\vartheta} \right\rangle_\vartheta , \\
q(t,s) &= \left\langle \langle n_\vartheta(t)\, n_\vartheta(s) \rangle_{\phi_\vartheta} \right\rangle_\vartheta , \\
G(t,s) &= \left\langle \frac{\delta \langle n_\vartheta(t) \rangle_{\phi_\vartheta}}{\delta \phi(s)} \right\rangle_\vartheta .
\end{aligned}
$$

- Inner averages over noise ϕ_ϑ evaluated using path-integral techniques (with an action that is a functional of m, q, and G).

Separation of Time Scales — Stationarity

- Assume macro-economic process $u_0(t)$ changes slowly: e.g.

$$\mathrm{d}u_0 = -\gamma u_0 \mathrm{d}t + \sqrt{2\gamma}\,\mathrm{d}W_0 \ , \qquad \gamma \ll 1 \ ,$$

  - ...so that the system becomes statistically stationary at given u_0
  - Derive FPEs for stationary states $\Rightarrow u_\vartheta$ OU process

$$\begin{aligned} m &= \Big\langle \Big\langle g(\overline{u}_\vartheta + \sigma_{u_\vartheta} x) \Big\rangle_x \Big\rangle_\vartheta , \\ q(\tau) &= \Big\langle \Big\langle g\big(\overline{u}_\vartheta + \sigma_{u_\vartheta} x\big)\ g\big(\overline{u}_\vartheta + \sigma_{u_\vartheta} y\big) \Big\rangle_{x\,y} \Big\rangle_\vartheta , \\ \chi &= \Big\langle \Big\langle g'(\overline{u}_\vartheta + \sigma_{u_\vartheta} x) \Big\rangle_x \Big\rangle_\vartheta , \\ \hat{C}(0) &= \int_{-\infty}^{+\infty} \mathrm{d}\tau \left[q(\tau) - q\right] , \end{aligned}$$

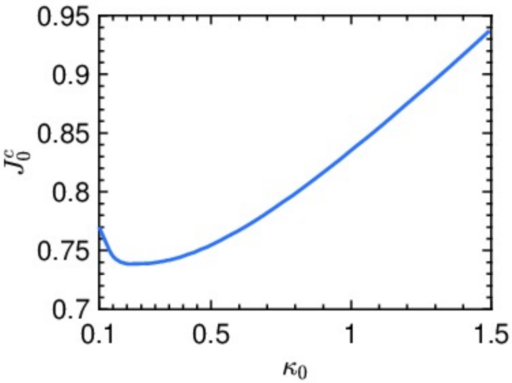
  - in which $q = \lim_{\tau\to\infty} q(\tau)$, and χ is an integrated response.

- For details, see K Anand, J Khedair, and RK, PRE **97** 052312 (2018).

Outline

Phase Structure

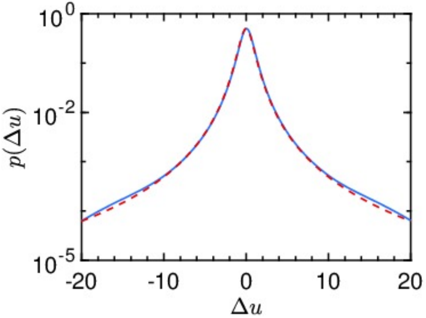
- System exhibits a glassy pase in large parts of parameter space (sufficiently small J_0/J, sufficiently small noise $\sigma_i \equiv \sigma$).



FM-SG boundary for $I \sim \mathcal{N}(0, \sigma_I^2)$, $u_0 = 0$; $J = 0.5$, $\alpha = 0.5$ $\sigma = 0.1$.

Return Distributions

- Distribution of returns

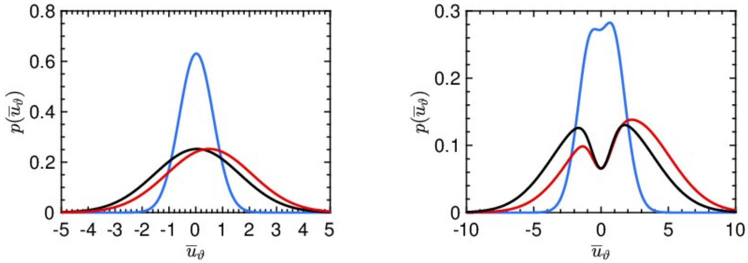


Distribution of returns for exponentially distributed κ with $\langle\kappa\rangle = \kappa_0 = 0.2$ and $\kappa_0|t - t'| = 20$.

$J_0 = J = \alpha = 0.5$, long time asymptotics (full line) and numerical evaluation (dashed), $\nu = 1$.

Collective Pricing

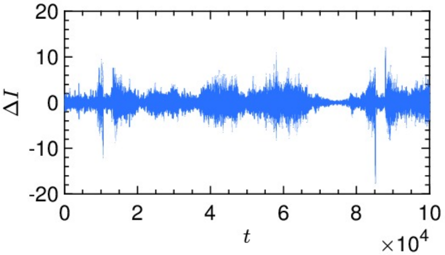
- Quasi-stationary equilibrium log-prices $\bar{u}_\vartheta$ determined by collective effects

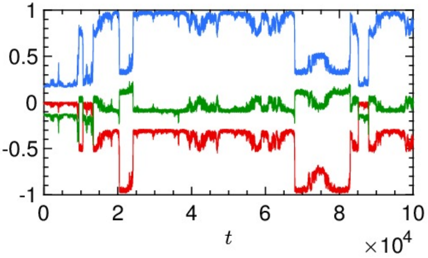


Distributions of equilibrium log-prices. Left: Non-interacting system Right: Interacting system. Narrow blue curves $\kappa = 0.5$, $u_0 = 0.1$, Wider set of curves: $\kappa = 0.2$ and $u_0 = 0.1$ for the nearly symmetric (black) curves; $u_0 = 0.5$ for the more asymetric (red) curves. Overall Γ distributed κ with $\nu = 1$ and. $\kappa_0 = 0.2$. Interacting system $J_0 = J = \alpha = 0.5$

Volatility Clustering and Metastability

- Embed attractors of known structure





$$J_{ij} \to J_{ij} + \frac{1}{N} \sum_{\mu=1}^{p} \xi_i^\mu \xi_j^\mu$$

$$m_\mu(t) = \frac{1}{N} \sum_i \xi_i^\mu \, g(u_i(t))$$

Top: changes of the market index for $\Delta t = 25$. Bottom: overlaps with three unbiased random patterns embedded in a system of $N = 50$ assets, with $\gamma = 10^{-4}$.

Outline

Inference — Simple ML Approach

- Use model to test inference algorithms and identify strengths/weaknesses
- In second step apply to real data (S&P 500)
- Log-likelihood (discretize time: Δ); parameters determined only by continuous part of trajectory.

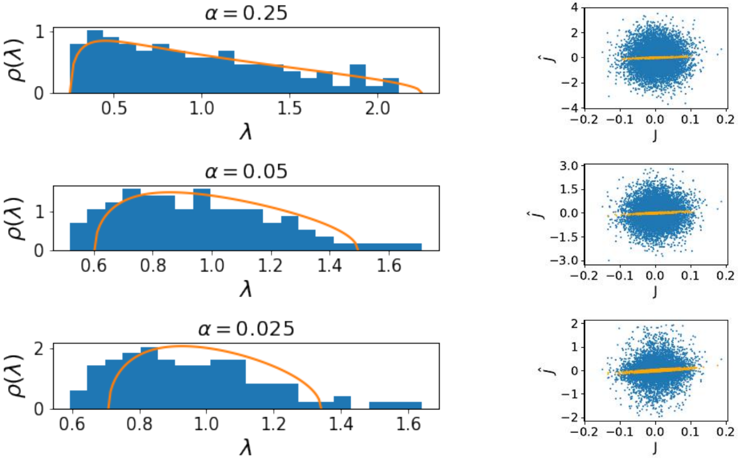
$$\mathcal{L} = \sum_{i,t} \frac{\Delta}{2\sigma_i^2} \Big[\dot{u}_i - f_i(\boldsymbol{u}(t))\Big]^2$$

  with

$$f_i(\boldsymbol{u}(t)) = -\kappa_i u_i + I_i + \sum_j J_{ij} g(u_j) + \sigma_0 u_0$$

  Parameters $\boldsymbol{\theta} = \{\kappa_i, I_i, J_{ij}\}$
- Use stochastic gradient descent or data batches to solve $\nabla_{\boldsymbol{\theta}} \mathcal{L} = 0$. Second method gives linear equations with coefficients determined by various sample-correlations.
- Issues: **(i)** sampling noise, **(ii)** non-ergodicity of the dynamics.

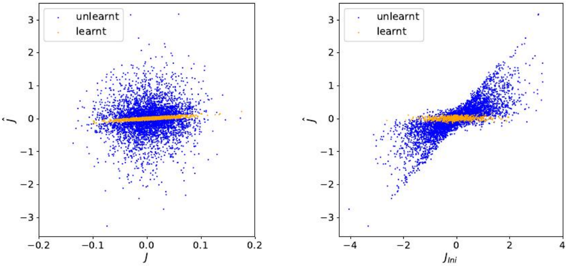
Issue (i): Sampling Noise — RMT



ML equations require inversions of various correlation matrices that are estimated, sampling noise $\Rightarrow$ random Matrices. Shown are (left) spectra of estimated correlation matrices $C_{ij} = \langle \delta g(u_i) \delta g(u_j) \rangle$, compared with Marčenko Pastur law, and (right) corresponding scatter-plots of $\hat{J}$ vs. J_{true}. Here $N = 125$, and $\alpha = N/T$

Issue (ii): Non-Ergodicity

- System dynamics is non-ergodic.
- Learning couplings requires to sample sufficiently many ergodic components
- For fixed data sample size this depends on ergodic time-scale γ^{-1}.



Scatter plots of estimated vs true couplings (Left), and plots of estimated vs initial couplings (Right) for a partially learnt situation. Parameters are $N = 150, T = 10^5$, and $\gamma = 10^{-7}$.

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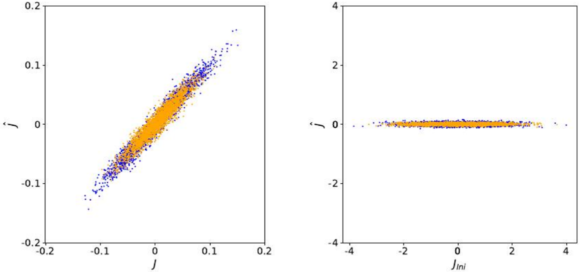

Scatter plots of estimated vs true couplings (Left), and plots of estimated vs initial couplings (Right) for a fully learnt situation.

Parameters are $N = 150, T = 10^5$, and $\gamma = 10^{-2}$.

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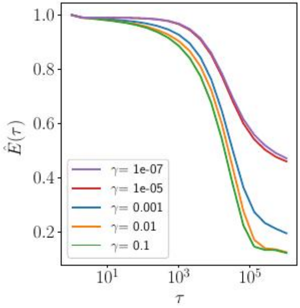


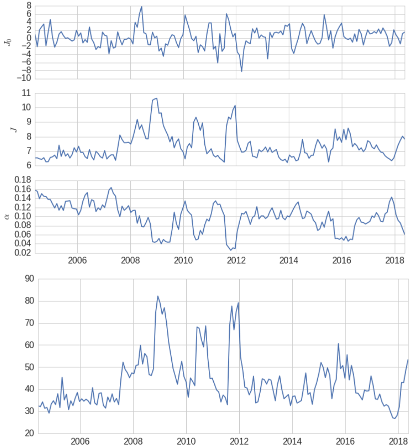
(Left): Normalized error of couplings in gradient descent learning as function of number of iterations for various γ.

Real-Data

- Lots of issues (splits, discontinued trading, out-listing)
- Use interacting model only on a 'co-moving' frame
- Analysis predicts significant levels of interaction
- Inferred model reproduces some global properties of real data, such as
    - return correlations
    - distributions of (log)-returns

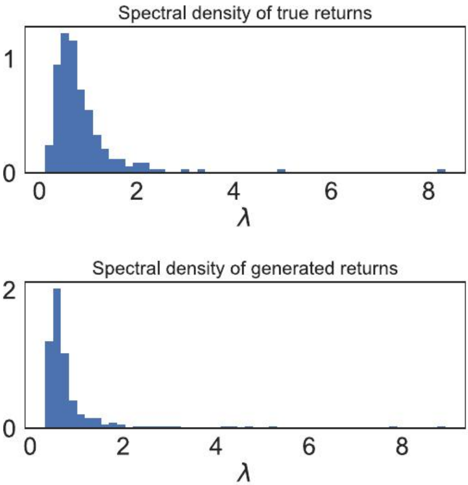
  with reasonable accuracy.
- Use for risk-analysis? Early Warning Indicators?

Real-Data: S&P 500 — Inferred Couplings



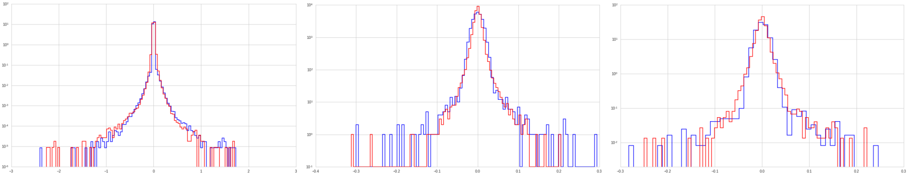
(Top) Mean, standard deviation and forward backward correlations between couplings as functions of time over a 14 year period starting in Jan 2004, re-evaluated every 30 days (based on data of preceeding 150 days). (Bottom: Top singular value of inferred coupling matrix. Data for $N = 200$ continuously listed S&P500 stocks.

Real-Data: S&P 500 — Return Correlations



Spectrum of correlation matrix $C_{ij} = \langle \delta u_i \delta u_j \rangle$ of true returns and of correlation matrix of S&P 500 log-returns generated (4 months in 2017) from model (inferred from 6 month of prior data). Parameters are $N = 200$, $N/T = 0.03$. (Note: Jumps not yet included in generative Model).

Real-Data: S&P 500 — Return Distributions



Distribution of true (red) and predicted (blue) 5-min log-returns across the market (Left) and for two randomly chosen assets (Middle & Right). Predictions are for 3 months ahead; statistics taken over 8 months, May–Dec 2016. Parameters are $N = 220$, $T = 10^4$. Jumps included in generative model.

Real-Data — Market States?

(Top): Overlap of market state with 3 selected singular vectors of the inferred interaction matrix as a function of time for a 5y period. (Bottom): Concurrent changes of the index. The period includes two major restructurings overlapping with the Draghi speech 26/07/12 and with the flash crash of 24/08/15.

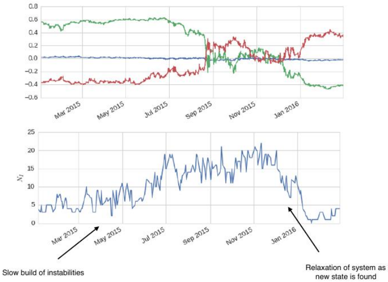
Real-Data — Detecting Instabilities?

- Assess stability of system trajectories of market

$$du_i(t) = f_i(\boldsymbol{u}_t)dt + \sigma_i dW_i(t)$$

by looking at eigenvalues of the Hessian

$$H_{ij} = \frac{\partial f_i(\boldsymbol{u}_t)}{\partial u_{jt}}$$



(Top): Overlap of market state with 3 selected singular vectors of the inferred interaction – zoom into period surrounding flash-crash. (Bottom): Concurrent evolution of the number of unstable directions of the system dynamics.

Outline

Summary

- Argued
  - that market model formulated in terms of asset prices should exhibit interactions between prices, which exhibit memory.
  - simplest interacting generalization of GBM has structure of a NN
- Expect generally many meta-stable phases.
- Different susceptibilities within phases entail different volatilities.
- Find key properties of market dynamics in (semi-)quantitative fashion.
- Fat tailed return distributions, non-trivial equilibrium pricing distributions
- Clear relation between volatilities and meta-stable states.
- Started inference (synthetic and real data)
  - issues of sampling noise and non-ergodicity
  - real data reasonably well reproduced by simple inferred model
  - of use for risk-management?

Thank You!

Return Distributions

- Compute distribution of returns

$$\Delta u_\vartheta \equiv u_\vartheta(t) - u_\vartheta(t')$$

in the quasi-stationary regime $\gamma|t-t'| \ll 1$.

- For individual u_ϑ find

$$\Delta u_\vartheta \sim \mathcal{N}\left(0, \frac{\sigma^2}{\kappa}\left(1 - \mathrm{e}^{-\kappa|t-t'|}\right)\right) .$$

- Time-scales (i) short: $\kappa|t-t'| \ll 1$, (ii) medium: $\kappa|t-t'| = \mathcal{O}(1)$, (iii) long: $\kappa|t-t'| \gg 1$.
- Assuming the κ are Γ- distributed

$$P(\kappa) = \frac{1}{\kappa_0 \Gamma(\nu)} \left(\frac{\kappa}{\kappa_0}\right)^{\nu-1} \exp(-\kappa/\kappa_0) ,$$

distribution of returns across the market (at long times $\kappa|t-t'| \gg 1$):

$$p(\Delta u) = \frac{\sqrt{\kappa_0}}{\sqrt{2\pi\sigma^2}} \frac{\Gamma(\nu + \frac{1}{2})}{\Gamma(\nu)} \left(1 + \frac{\kappa_0(\Delta u)^2}{2\sigma^2}\right)^{-(\nu+1/2)} .$$

$\Rightarrow$ fat power-law tails.