Mellin Moments of the Unpolarized Gluon PDF in the Proton from Nonlocal Operators

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Motivation

- Ultimate goal of understanding full x-dependence
  - Large errors obscure tension with phenomenology
  - Reconstruction includes model dependence

$$a_n(\mu) = \int_0^1 dx\, x^{n-1}\, g(x, \mu)$$

- Moments provide insight to relative behaviors in extreme-x regions in simplified manner
  - Allow for careful study of systematics
- Can be directly compared with global analysis results and provide further constraints for synergistic studies

OPE of the Gluon Operator

- We extract matrix elements of non-local operators on the lattice

$$M_{\mu i;\nu j}(P,z) = \langle N(P)\,|\,\mathrm{Tr}\big[F_{\mu i}(z)W(z,0)F_{\nu j}(0)W(0,z)\big]\,|\,N(P)\rangle$$

- At small spatial separations, this can be expanded through the OPE as

$$F_{\mu i}(z)W(z,0)F_{\nu j}(0)W(0,z) = \sum_{n=0}^{\infty} \frac{z^{\rho_1}\ldots z^{\rho_n}}{n!} C_n(z^2,\mu^2)\mathcal{O}^{g,(n)}_{\mu i;\nu j,\{\rho\}}(\mu)$$

  - Where C_n are perturbatively calculable Wilson coefficients and $\mathcal{O}^{g,(n)}_{\mu i;\nu j,\{\rho\}}(\mu)$ are symmetric, traceless twist-two gluon operators

- The matrix elements of these operators with respect to boosted nucleon states are related to moments of the gluon PDF $a_n(\mu) = \int_0^1 dx x^{n-1} g(x,\mu)$

$$\langle N(P)\,|\,\mathcal{O}^{g,(n)}_{\mu i;\nu j,\{\rho\}}(\mu)\,|\,N(P)\rangle = 2P^{\rho_1}\ldots P^{\rho_n} a^g_{n+2}(\mu)$$

- Enforcing collinear kinematics results in

$$M(P,z) = \sum_{n=0}^{\infty} \frac{(i\nu)^n}{n!} C_n(z^2,\mu^2)\, a^g_{n+2}(\mu)$$

Moments from Short-Distance Factorization

- An equivalent formulation comes from the Ioffe-time distribution (ITD), which is related to the PDF by a cosine Fourier transform

$$Q_{g/S}(\nu, \mu^2) = \int_0^1 dx\ \cos(x\nu)\, x f_{g/S}(x, \mu^2) = \int_0^1 dx \sum_{n=0}^{\infty} (-1)^n \frac{(x\nu)^{2n}}{(2n)!}\, x f_{g/S}(x, \mu^2)$$

- In SDF, the ITD can be extracted from lattice MEs by a perturbatively calculable matching relation

$$\mathfrak{M}(\nu, z^2) Q_g(0,\mu^2) = Q_g(\nu, \mu^2) - \frac{\alpha_s N_c}{2\pi} \int_0^1 du\, Q_g(u\nu, \mu^2) \left[\ln\left(\frac{z^2 \mu^2 e^{2\gamma_E}}{4} \right) \mathfrak{B}_{gg}(u) + L(u) \right]$$
$$- \frac{\alpha_s C_F}{2\pi} \ln\left(\frac{z^2 \mu^2 e^{2\gamma_E}}{4} \right) \int_0^1 dw \left[Q_S(w\nu, \mu^2) - Q_S(0,\mu^2) \right]$$

with

$$\mathfrak{M}_g(\nu, z^2) \equiv \left(\frac{M_g(\nu, z^2)}{M_g(\nu,0)\,|_{z=0}} \right) \Big/ \left(\frac{M_g(0,z^2)\,|_{p=0}}{M_g(0,0)\,|_{p=0,z=0}} \right)$$

- Integrating the matching kernel provides Wilson coefficients

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- Integrating the matching kernel provides Wilson coefficients

ITD serves as a generator of moments

$$Q_g(\nu,\mu^2) = 1/2\int_{-1}^1 dx\, e^{ix\nu} x f_g(x,\mu^2) = \int_0^1 dx \sum_{n=0}^{\infty}\frac{(ix\nu)^n}{n!} x f_g(x) \longrightarrow \mathfrak{M}_g(\nu,z^2) = 1 - \sum_{n=2}^{\infty}\frac{(i\nu)^n}{n!}C_n(z,\mu)\frac{\langle x^{n+1}\rangle}{\langle x\rangle}$$

Numerical Approach

- We extract moments from the gluon reduced-ITD (neglecting the singlet contribution) according to a truncated form of the OPE

$$\mathfrak{M}_g(\nu, z^2) - 1 = \sum_{n=1}^{\frac{n_{max}}{2}} \widetilde{C}_n^g(z^2, \mu^2) \frac{\langle x^{2n+1} \rangle_g^{\mu^2}}{\langle x \rangle_g^{\mu^2}}$$

- We study various systematics within the fit
  - Truncation of the sum: $n_{max} \in (2, 4, 6)$
  - Spatial separations entering the fit (z_{min}, z_{max})
- For a given fit, we arrange the data as a vector of matrix elements (with i labelling pairs of (P, z))

$$y_i = \mathfrak{M}_g(\nu_i, z_i^2) - 1$$

- Which we can solve as a linear system $\quad y_i = \sum_j A_{ij} r_j \qquad r_j \equiv \frac{\langle x^{2j+1} \rangle_g^{\mu^2}}{\langle x \rangle_g^{\mu^2}}$
  - Perform a weighted least squares fit using the bootstrap uncertainties associated with the reduced-ITD

Lattice Setup

- Gluonic matrix elements extracted from operators of the form

$$M_{\mu i;\nu j}(P, z) = \langle N(P) \,|\, F_{\mu i}(z) W(z,0) F_{\nu j}(0) W(0,z) \,|\, N(P) \rangle$$

- Specific operator choice avoids finite mixing under renormalization

- Results shown for one ensemble of maximally twisted-mass fermions with clover improvement and the Iwasaki-improved gauge action

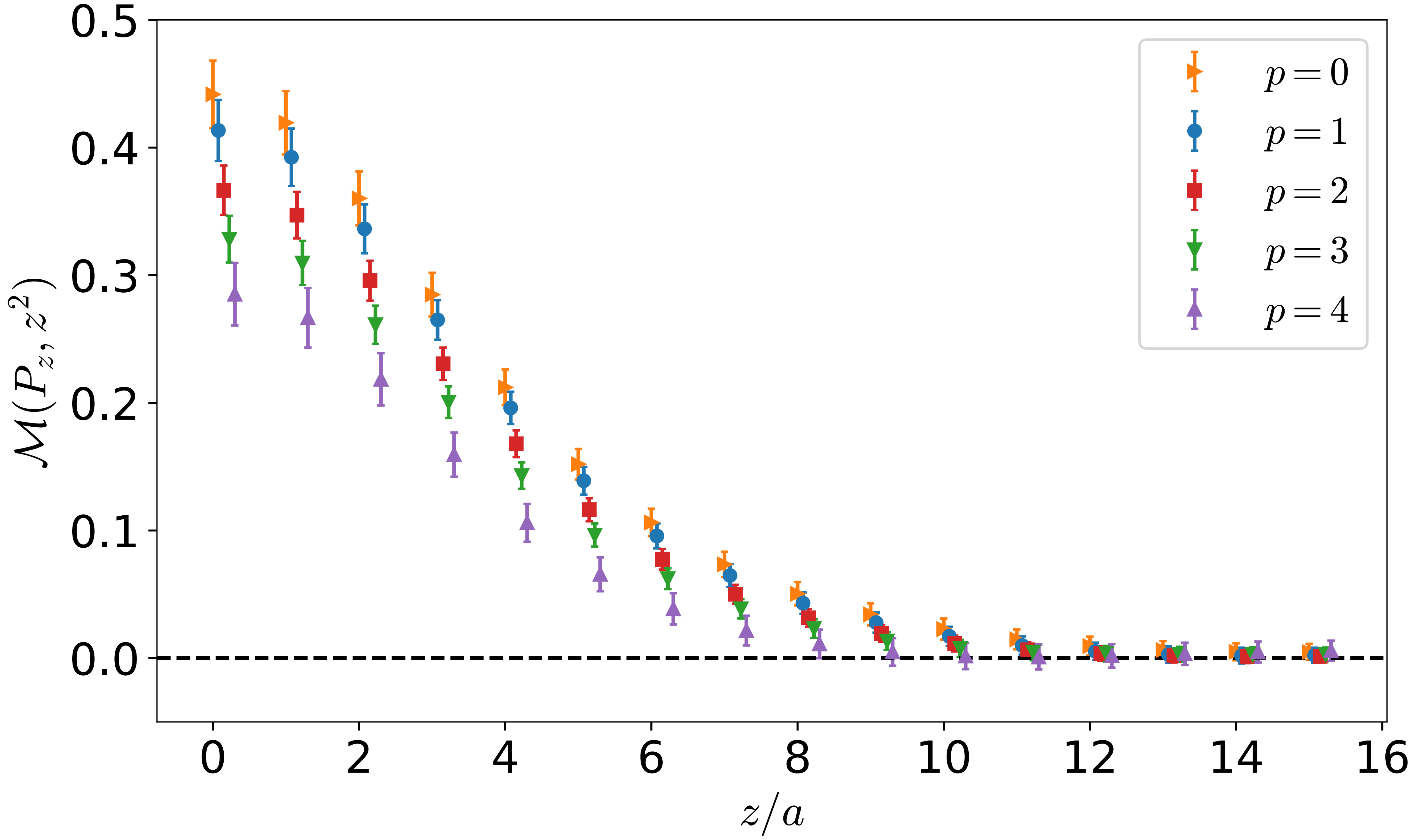
Parameters							
Ensemble	β	a [fm]	volume $L^3 \times T$	N_f	m_π [MeV]	Lm_π	L [fm]
cA211.30.32	1.726	0.0908	$32^3 \times 64$	2+1+1	260	4	3.0

- Gluonic quantities are very sensitive (purely disconnected contributions), requiring much higher statistics than quark case

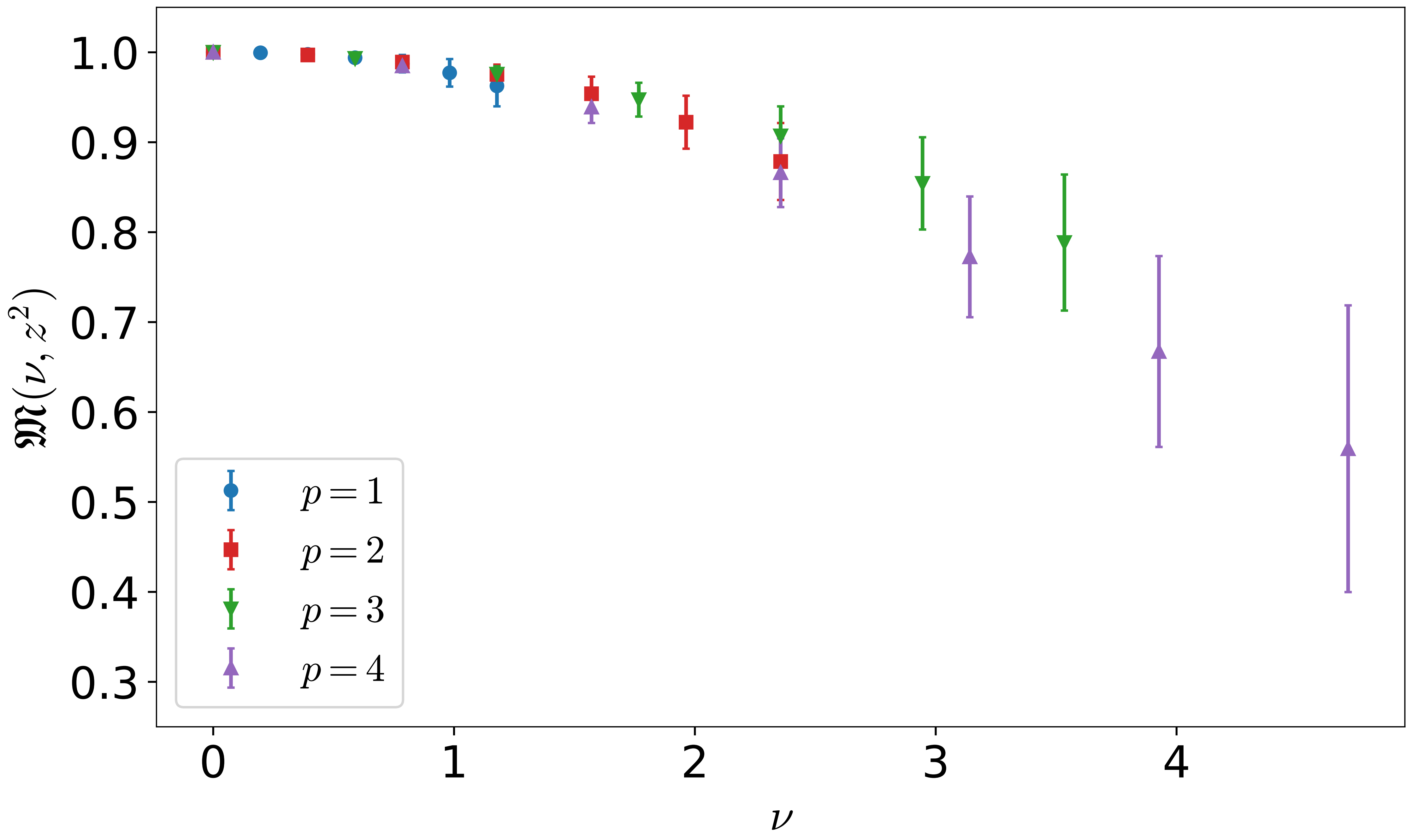
- Momentum smearing applied to $P > \frac{2\pi}{L}$

P [GeV]	N_{confs}	N_{src}	N_{dir}	N_{meas}
0, 0.43, 0.85, 1.28, 1.71	1,134	200	6	1,360,800

Lattice Results



[**JD** et al., Phys. Rev. D 108 9, 094515 (2023)]



- Results use (20, 10) steps of stout smearing on $F_{\mu\nu}$ and W respectively (4D, 4D)
- Signal has increased noise at larger values of P and z
- Different (P, z) pairs follow common ν dependence

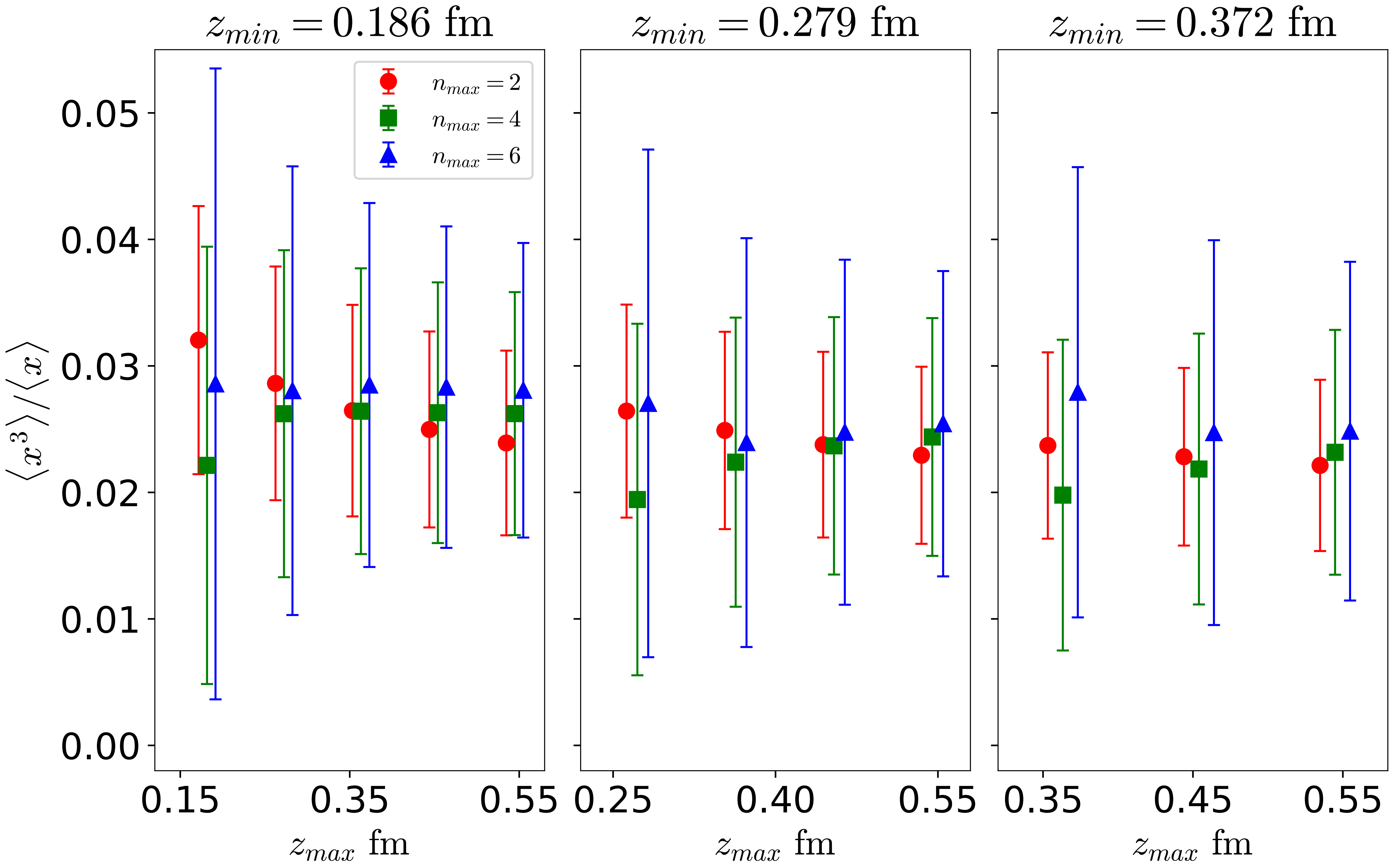
Moments at Fixed Scale

- **Moments are extracted directly at a scale $\mu = 2\,\mathrm{GeV}$**

- **Fixed-order results allow us to explore the stability of the fits for different choices of systematics**

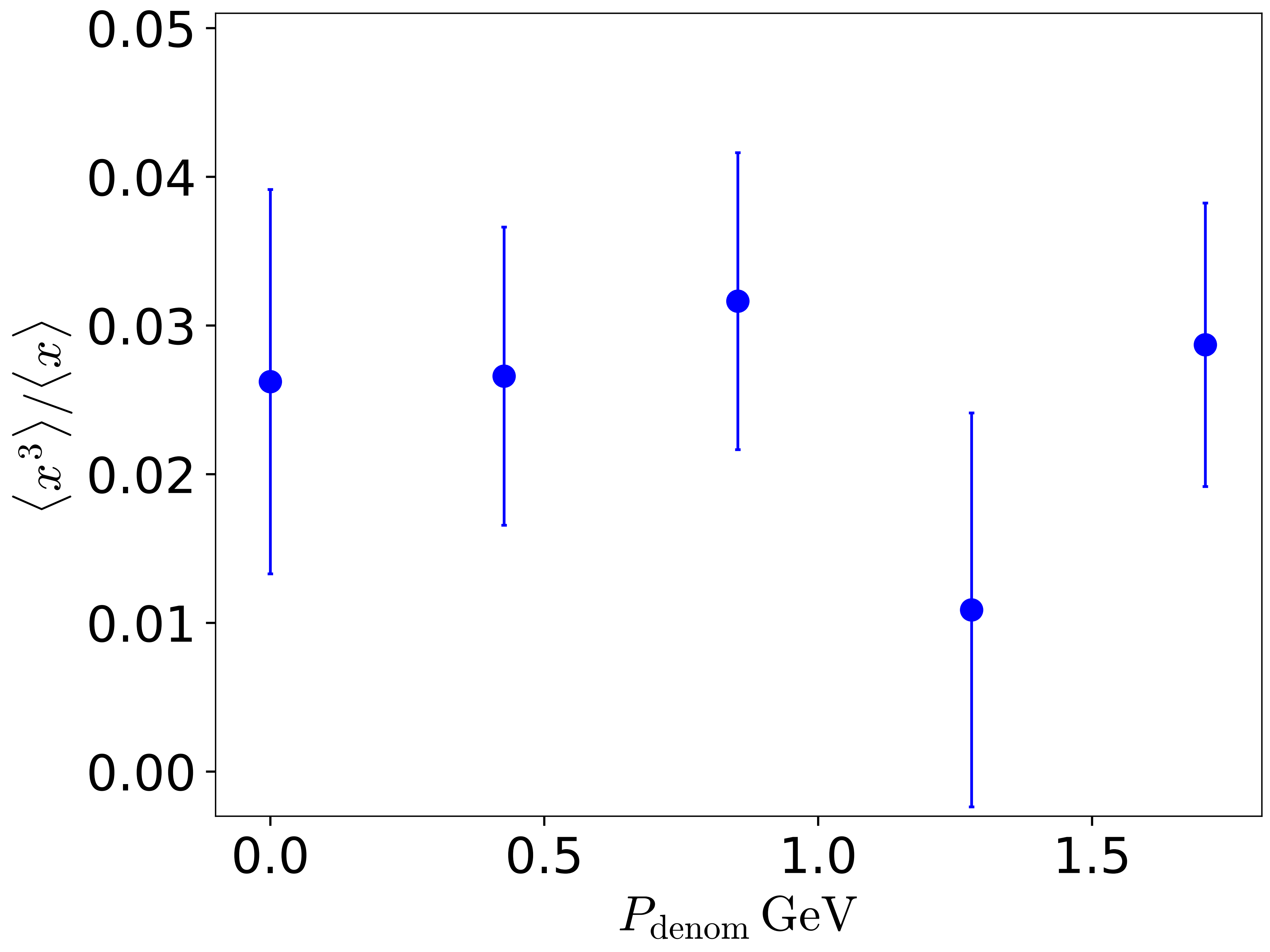
- **We find $\langle x^5\rangle/\langle x\rangle$ and $\langle x^7\rangle/\langle x\rangle$ are not well constrained**

- **Uncertainties generally correlated with degrees of freedom of the fit**



Mellin Moments from Alternative Ratios

- Moments can be extracted from ratios of $\mathfrak{M}(\nu, z^2)$ at different nonzero values of P [Phys. Rev. D 106, 074505 (2022)]
  - Potentially suppresses correlations between MEs, reducing statistical uncertainties
  - May partially suppress residual higher-twist effects associated with the leading-twist expansion
- Taking ratios of reduced-ITDs makes the problem non-linear
  - Fit moments through direct minimization of weighted χ^2



- We observe consistency within uncertainties for all choices of P_{denom}
  - Fit not well constrained for second largest value of P

RG Invariance of Moments

- Examining the evolution from initial scale μ_0 establishes uncertainties associated with perturbative effects
  - Moments evolved to $\mu = 2\,\mathrm{GeV}$ (taking $\alpha_s(2\,\mathrm{GeV}) = 0.293$) for comparison

- Moments evolve according to DGLAP equations

$$\frac{d}{d\ln(\mu^2)}\begin{pmatrix}\langle x^n\rangle_S^{\mu^2}\\ \langle x^n\rangle_g^{\mu^2}\end{pmatrix} = \begin{pmatrix}\gamma^{qq} & \gamma^{qg}\\ \gamma^{gq} & \gamma^{gg}\end{pmatrix}\begin{pmatrix}\langle x^n\rangle_S^{\mu^2}\\ \langle x^n\rangle_g^{\mu^2}\end{pmatrix} \qquad \gamma_n^{ij}(\mu^2) = \frac{\alpha_s(\mu^2)}{2\pi}\gamma_n^{ij,(0)} + \left(\frac{\alpha_s(\mu^2)}{2\pi}\right)^2\gamma_n^{ij,(1)}, \qquad i,j\in\{q,g\}$$

- We vary the initial scale in the matching through $\kappa = \left(\frac{1}{\sqrt{2}}, 1, \sqrt{2}\right)$

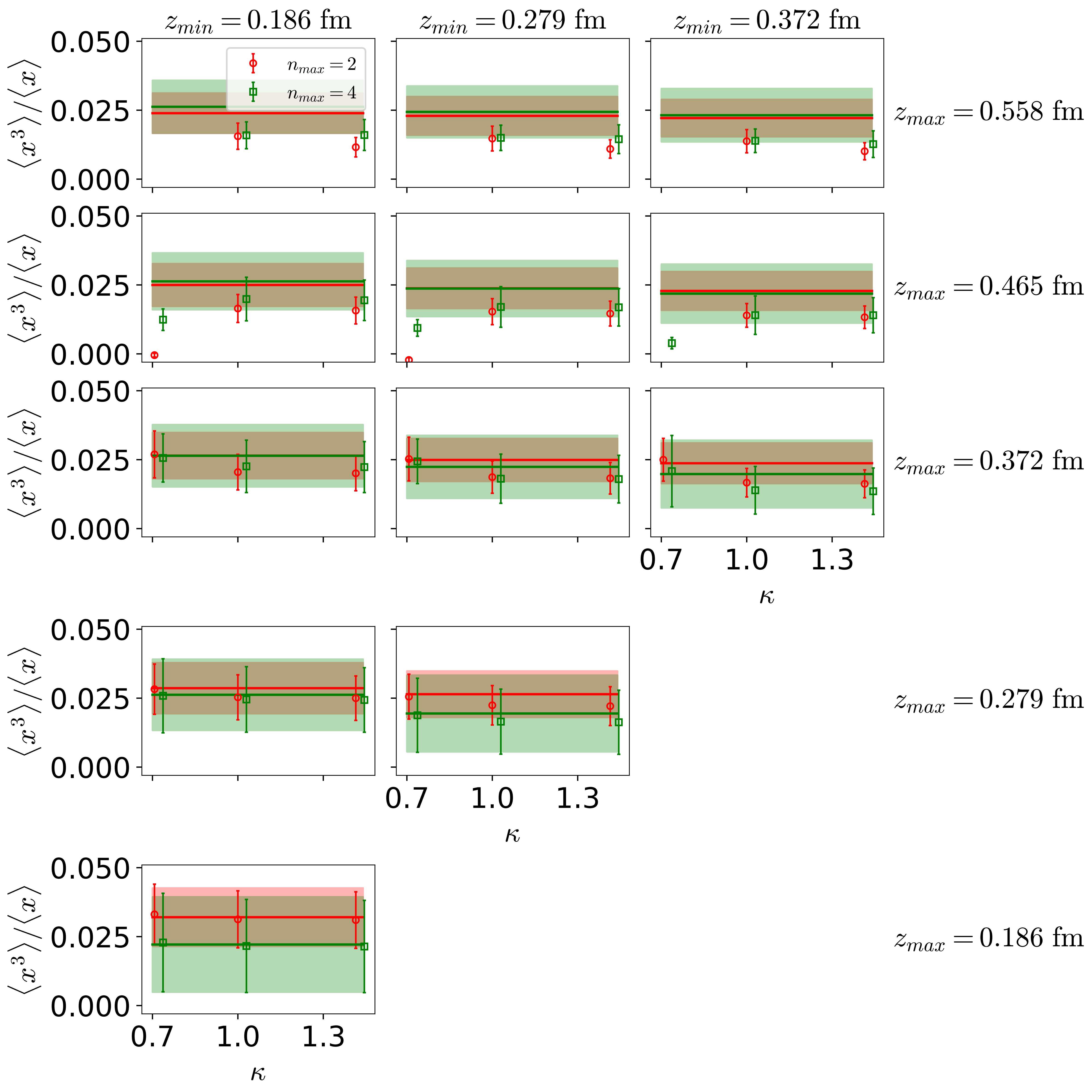
$$\kappa \propto \mu_0^2$$

$$\mathfrak{M}(\nu,z^2)Q_g(0,\mu^2) = Q_g(\nu,\mu^2) - \frac{\alpha_s N_c}{2\pi}\int_0^1 du\, Q_g(u\nu,\mu^2)\left[\ln\left(\frac{z^2\mu^2e^{2\gamma_E}}{4}\right)\mathfrak{B}_{gg}(u) + L(u)\right]$$

Evolution of Moments

- DGLAP carried out with two-loop matching (same for $\alpha_s(\mu)$ running)

- Missing or zero-valued points indicate unstable extraction due to breakdown of perturbative expansion



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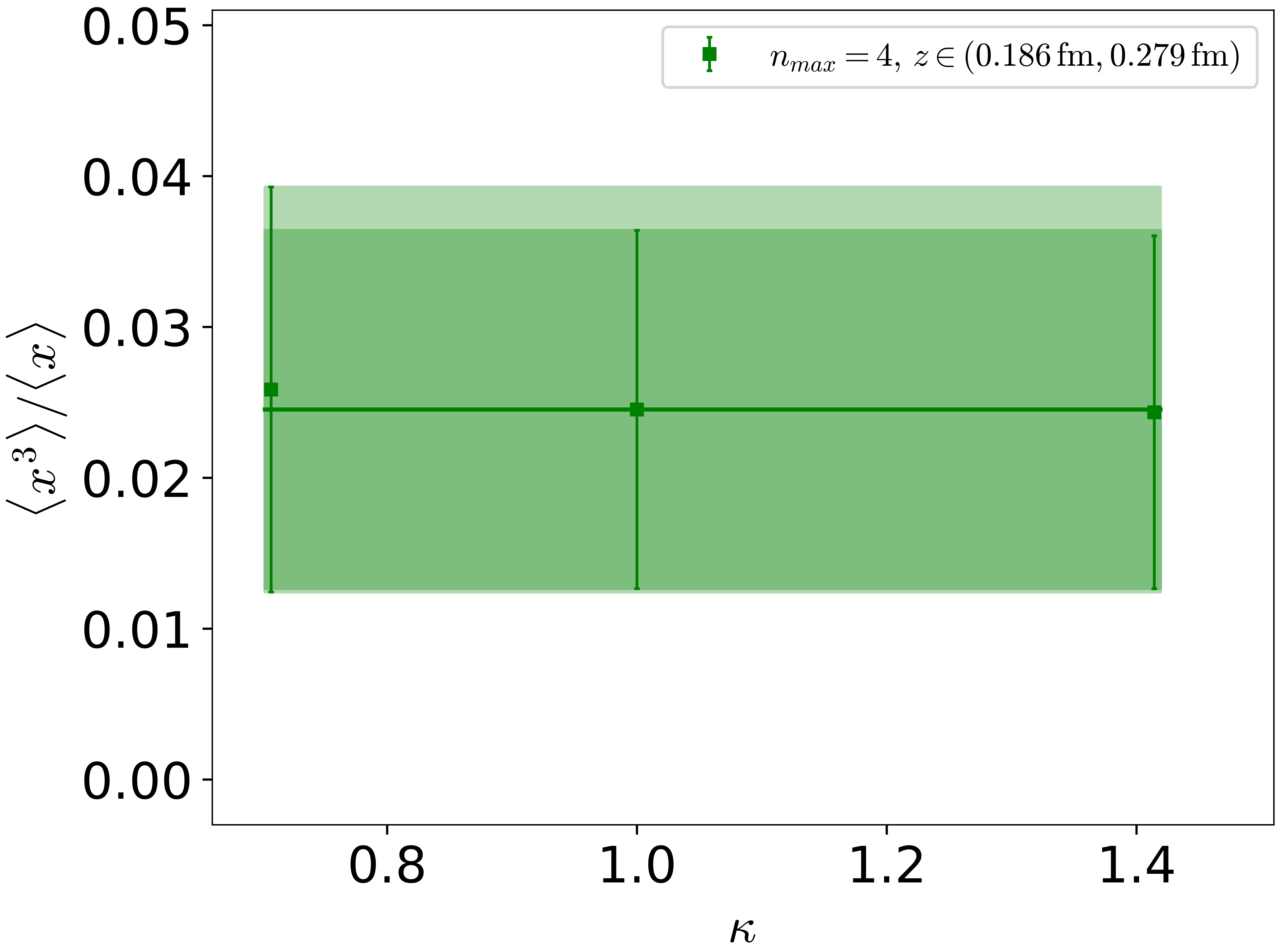
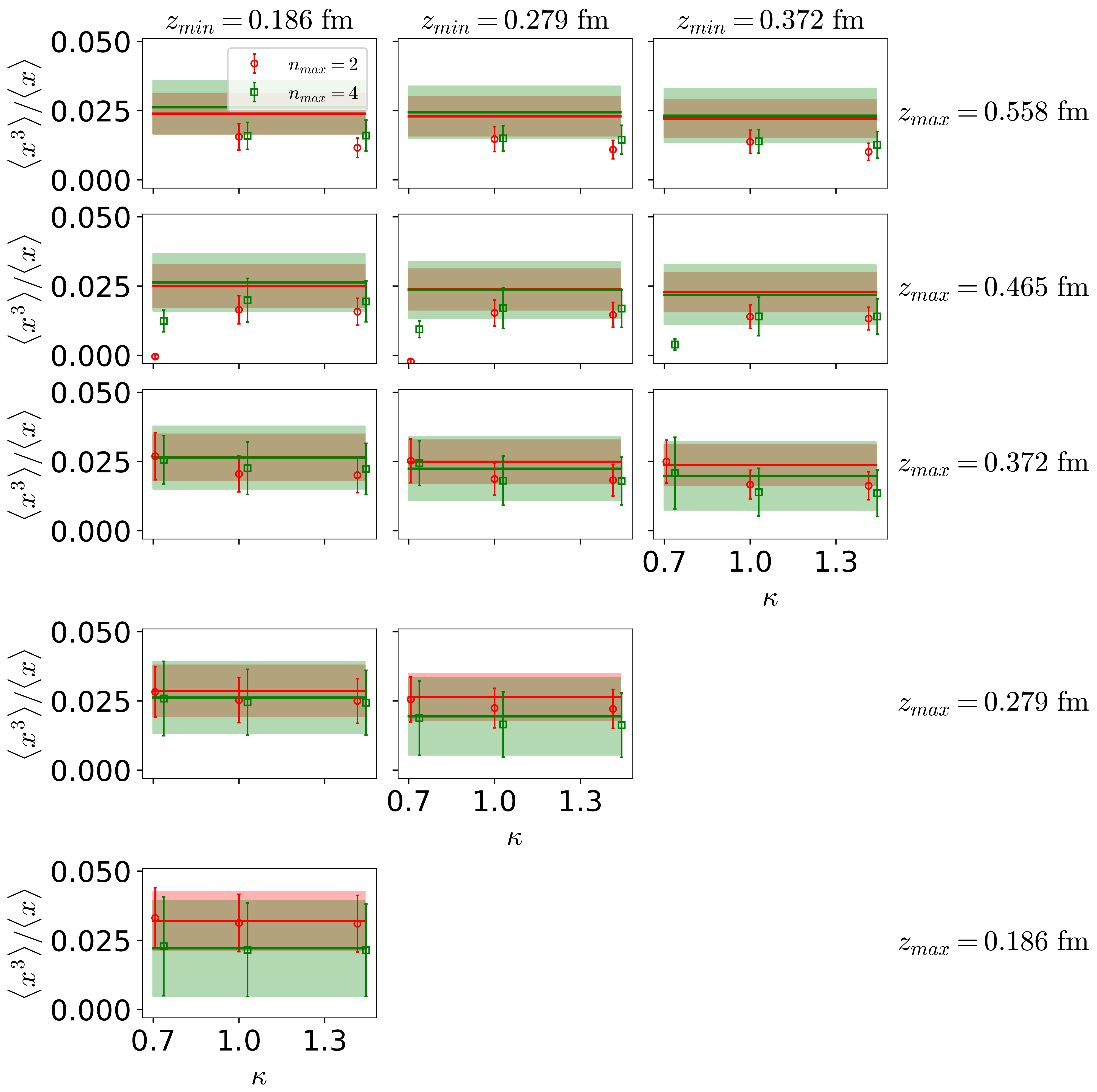




Mixing with Quark Singlet

- We incorporate the effect of the quark singlet by utilizing moments extracted through global analysis results (NNPDF4.0 and JAM)

$$\mathfrak{M}(\nu, z^2)Q_g(0,\mu^2) = Q_g(\nu, \mu^2) - \frac{\alpha_s N_c}{2\pi}\int_0^1 du\, Q_g(u\nu, \mu^2)\left[\ln\left(\frac{z^2\mu^2 e^{2\gamma_E}}{4}\right)\mathfrak{B}_{gg}(u) + L(u)\right]$$
$$-\frac{\alpha_s C_F}{2\pi}\ln\left(\frac{z^2\mu^2 e^{2\gamma_E}}{4}\right)\int_0^1 dw\left[Q_S(w\nu, \mu^2) - Q_S(0,\mu^2)\right]$$

- Singlet moments appear as an additional sum expansion

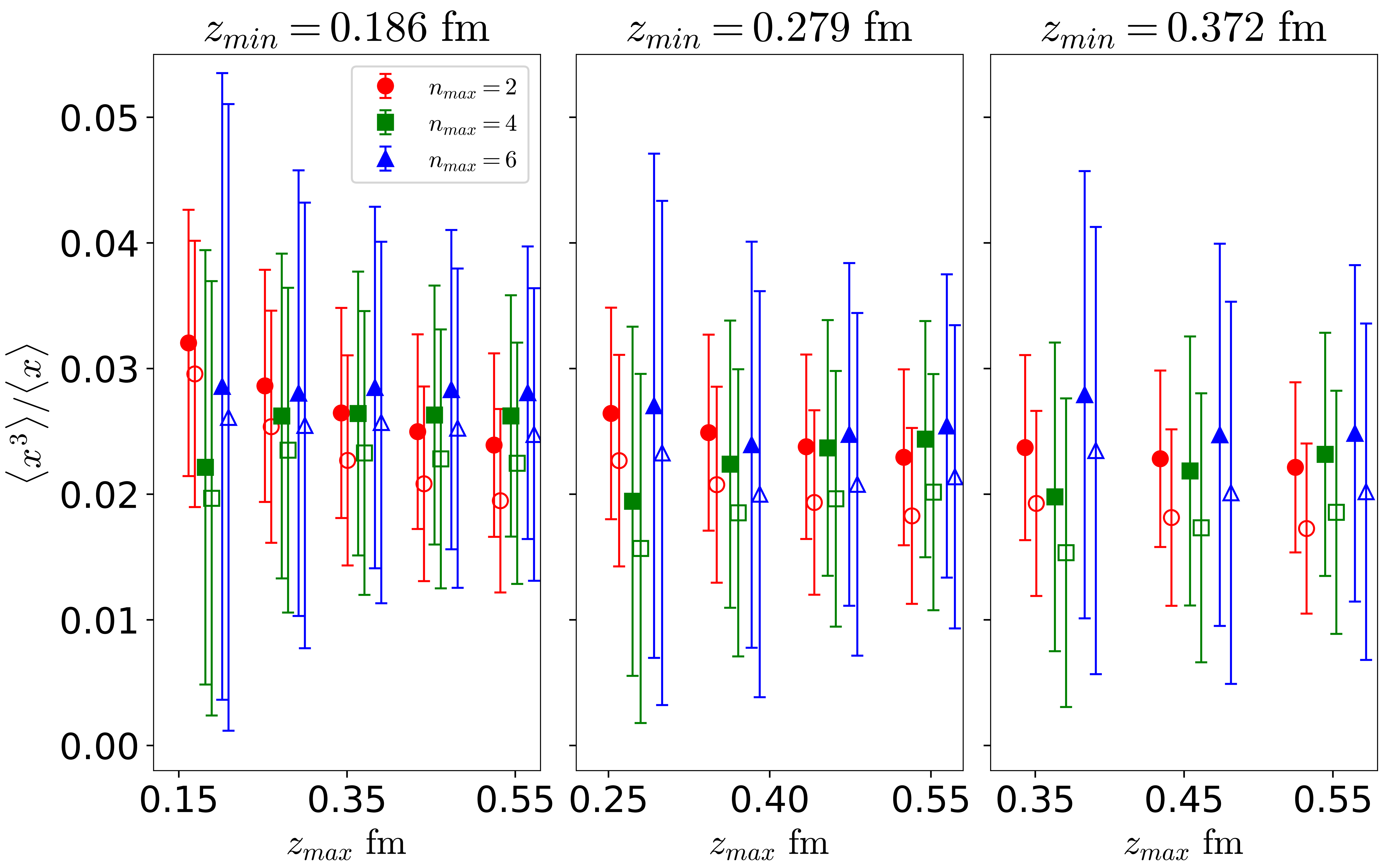
$$\mathfrak{M}_g(\nu, z^2) = 1 + \sum_{n=1}^{\infty}\tilde{C}_n^g(z^2, \mu^2)\frac{\langle x^{2n+1}\rangle_g^{\mu^2}}{\langle x\rangle_g^{\mu^2}} - \frac{\alpha_s C_F}{2\pi}\sum_{n=1}^{\infty}\tilde{C}_n^S(z^2, \mu^2)\frac{\langle x^{2n+1}\rangle_S^{\mu^2}}{\langle x\rangle_g^{\mu^2}}$$

- Utilizing numeric results from global analysis keeps the problem linear

- Moments mix order-by-order so we truncate at the same order in both sums

Fixed-Order Singlet Mixing

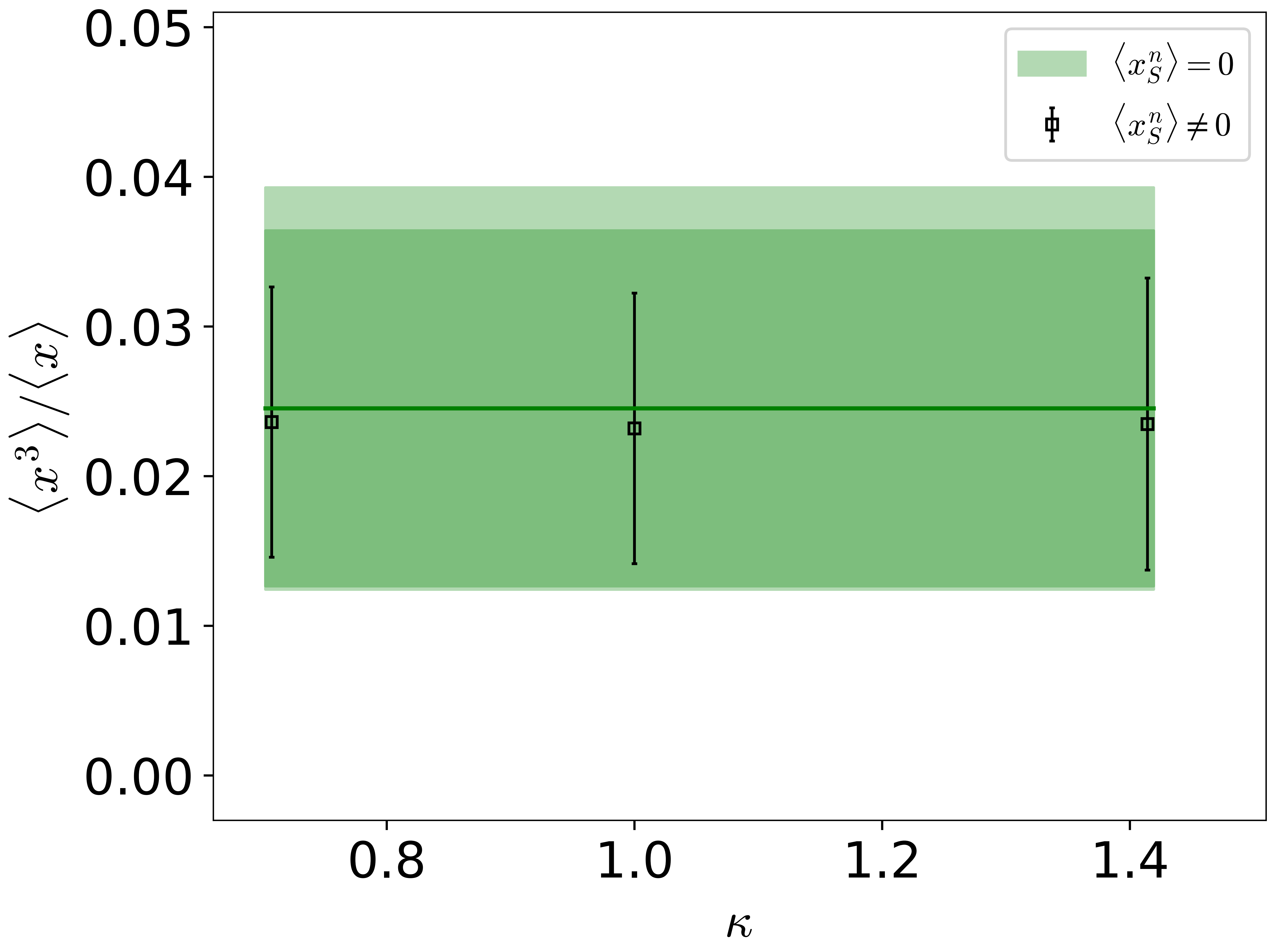
- Including the singlet contribution leads to a consistent downward shift at fixed-order
- Size of shift indicates the mixing effect is sub-leading
- Suggests expectation of lattice singlet affect should be of the same order as global analysis



Final Results

- Evolution of with singlet requires full evolution mixing matrix

$$\frac{d}{d\ln(\mu^2)}\begin{pmatrix}\langle x^n\rangle_S^{\mu^2}\\ \langle x^n\rangle_g^{\mu^2}\end{pmatrix} = \begin{pmatrix}\gamma^{qq} & \gamma^{qg}\\ \gamma^{gq} & \gamma^{gg}\end{pmatrix}\begin{pmatrix}\langle x^n\rangle_S^{\mu^2}\\ \langle x^n\rangle_g^{\mu^2}\end{pmatrix}$$



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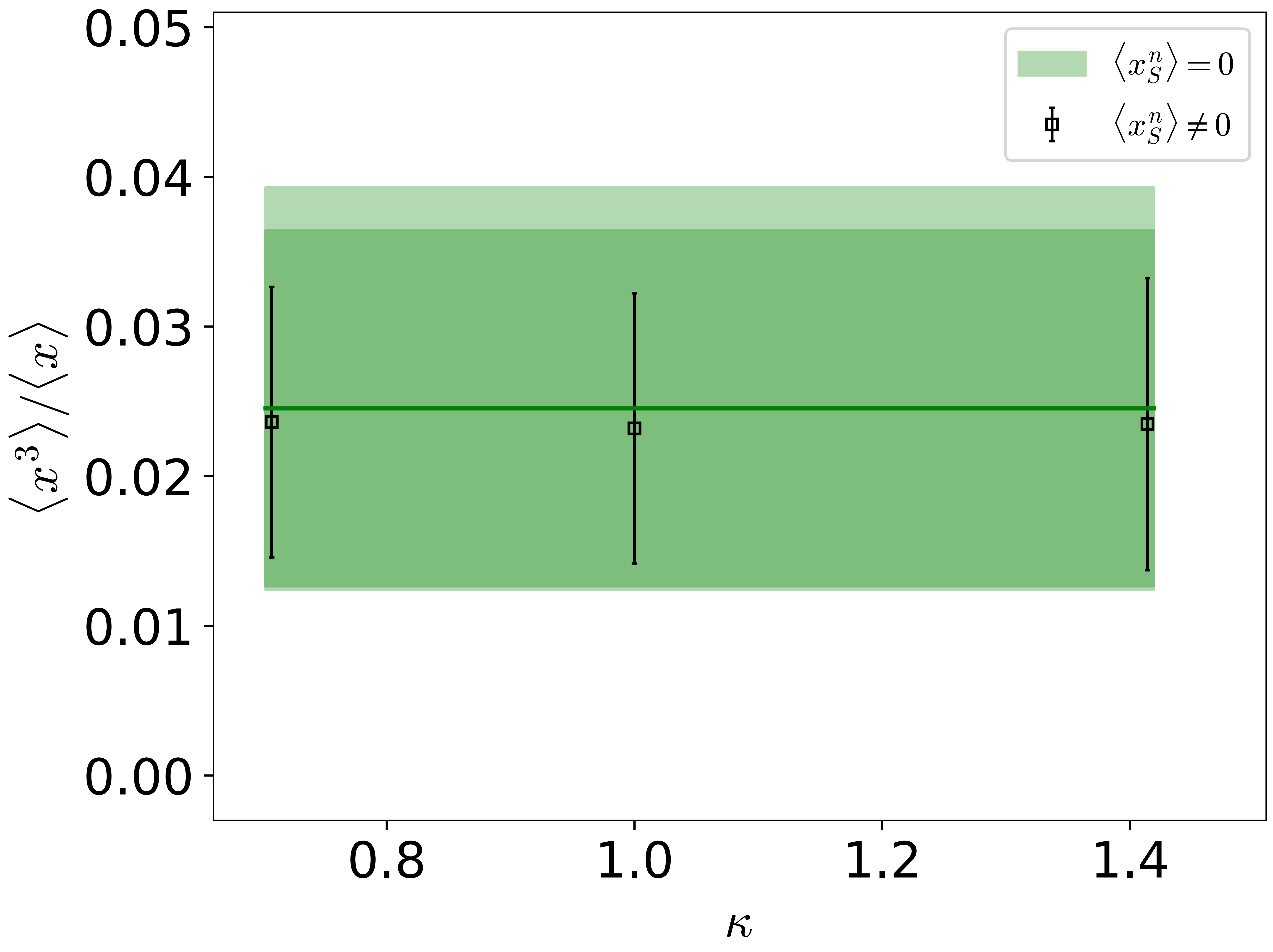
Analysis	$\langle x^3\rangle_g/\langle x\rangle_g$	$\langle x\rangle_g$
Lattice QCD, no mixing incorporated (this work)	$0.0250(120)^{+0.0029}_{-0.0002}$	—
Lattice QCD, mixing incorporated (this work)	$0.0232(90)^{+0.0010}_{-0.0004}$	—
NNPDF4.0 [Eur. Phys. J. C 82 5, 428 (2022)]	0.0281(11)	0.400(3)
JAM (NLO) [Phys. Rev. D 112, 114017 (2025)]	0.0317(17)	0.400(6)



- Evolved singlet result contained within gluon only result
- No significant tension between our result and global analysis at current precision

Thank You!