Extracting Vector Mellin Moments for the Pion and Kaon in LQCD

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Work with:
J. Torsiello (Temple), **K. Cichy** (Adam Mickiewicz University), **M. Constantinou** (Temple), **J. Delmar** (ANL)

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Outline

- ❖ Theoretical Formulation
- ❖ Lattice Formulation (focus of this talk)
  - ❖ Background
  - ❖ Lattice Methodology
  - ❖ Results
    - Matrix Elements
    - Double Ratio
    - Fixed z Analysis and combined Analysis
    - DGLAP Evolution
    - Results Comparison
    - SU(3) Symmetry Breaking
- ❖ Summary and Future Work

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Pion and Kaon PDFs from Lattice QCD via Large Momentum Effective Theory and Short-Distance Factorization

Joshua Miller,[1,*] Joseph Torsiello,[1,†] Isaac Anderson,[2,1] Krzysztof Cichy,[3] Martha Constantinou,[1,‡] Joseph Delmar,[1] and Sarah Lampreich[1]

Mellin Moments of Pion and Kaon Unpolarized PDFs from Nonlocal Operators in Lattice QCD

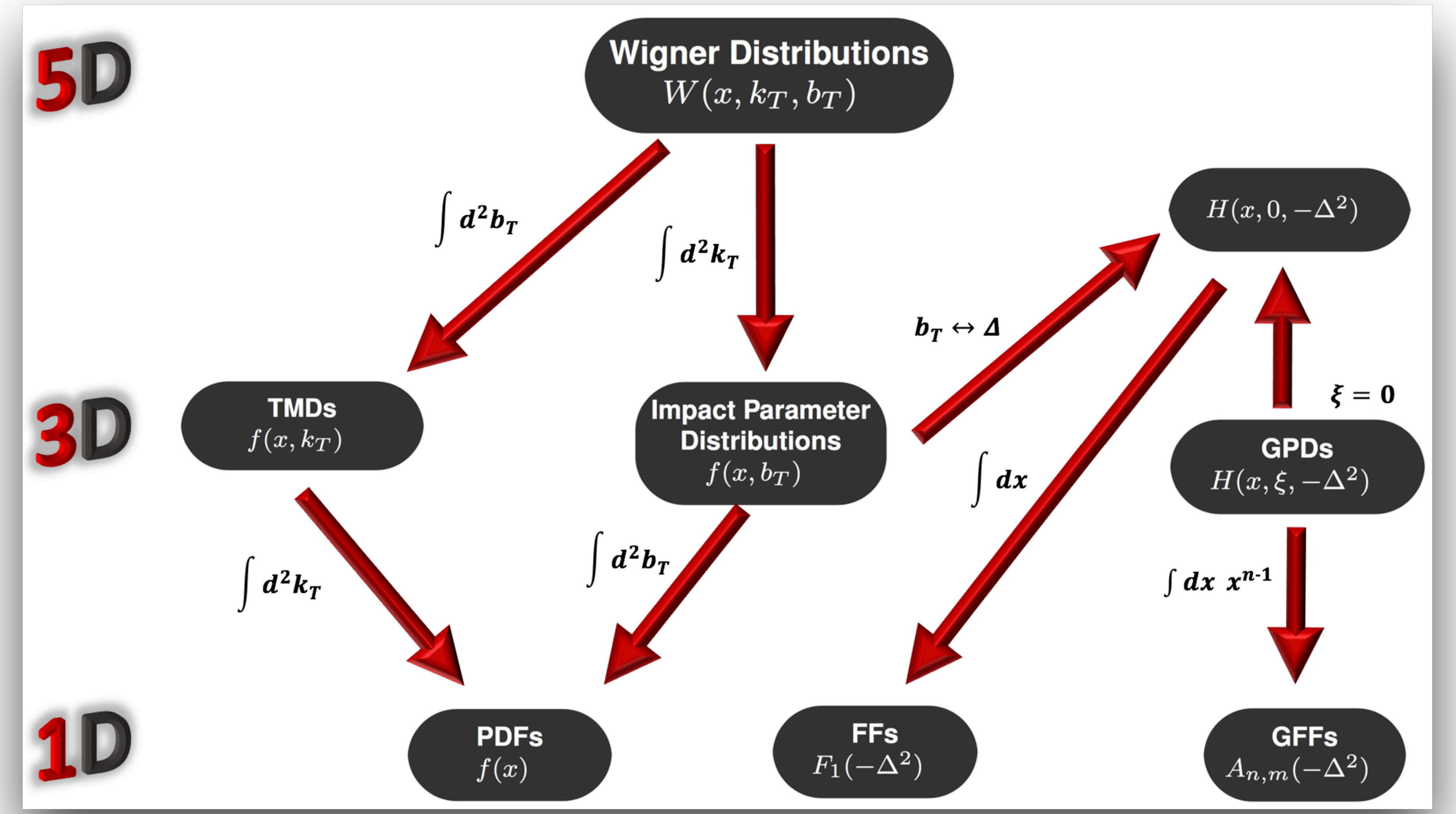
Joshua Miller,[1,*] Joseph Torsiello,[1] Krzysztof Cichy,[2] Martha Constantinou,[1,†] and Joseph Delmar[1,3]

arXiv:2606.28102 [hep-lat]

Parton Distribution Functions

❖ **PDFs are 1D:**

- Describe the probability a quark or gluon carry a fraction x of the hadron's longitudinal momentum



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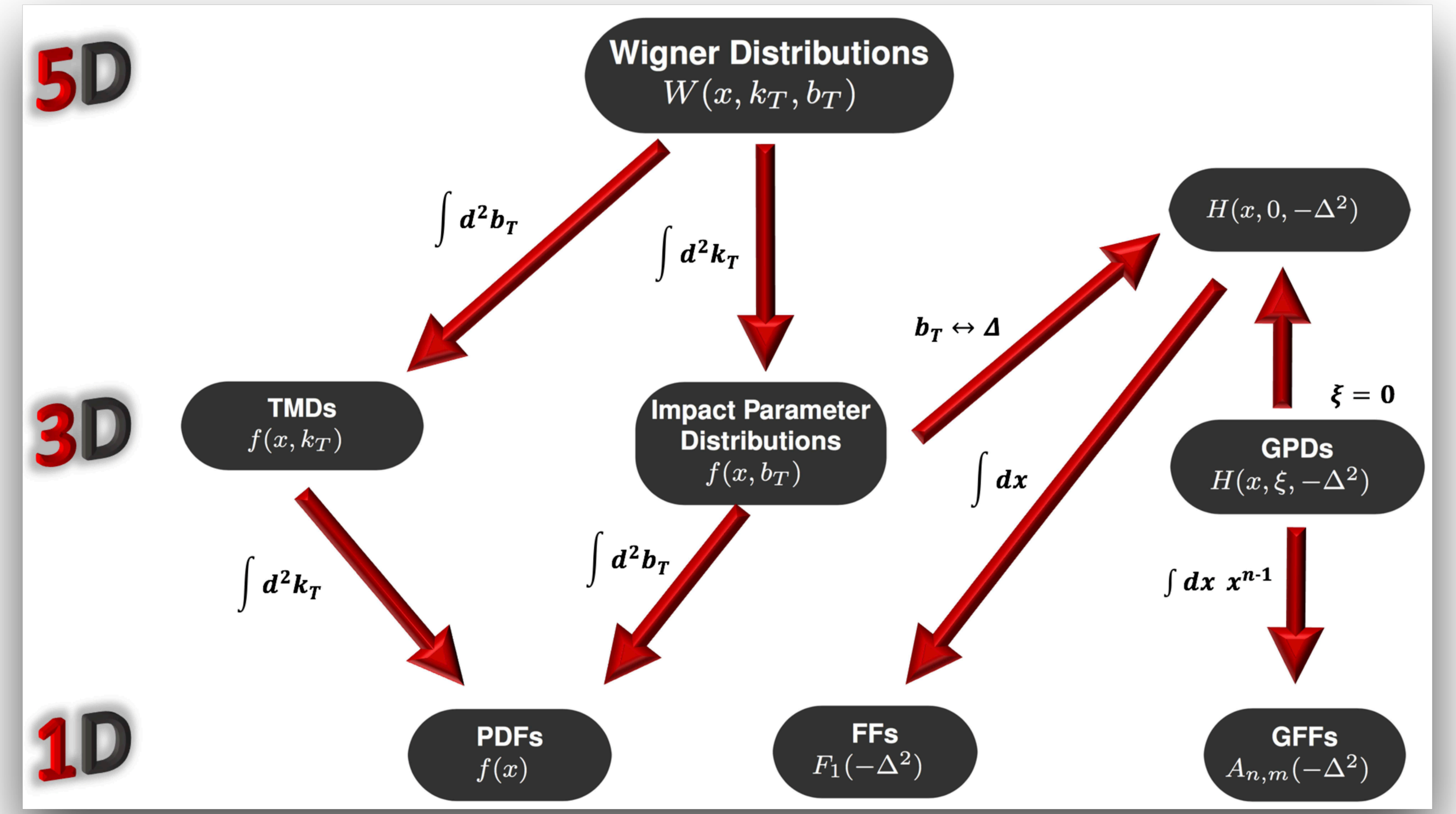
- **Experimentally constrained for nucleon…**
  - … but meson PDFs are relatively unknown
    - Pion information is extracted from pion-induced Drell-Yan experiments
    - Kaon information is even more limited



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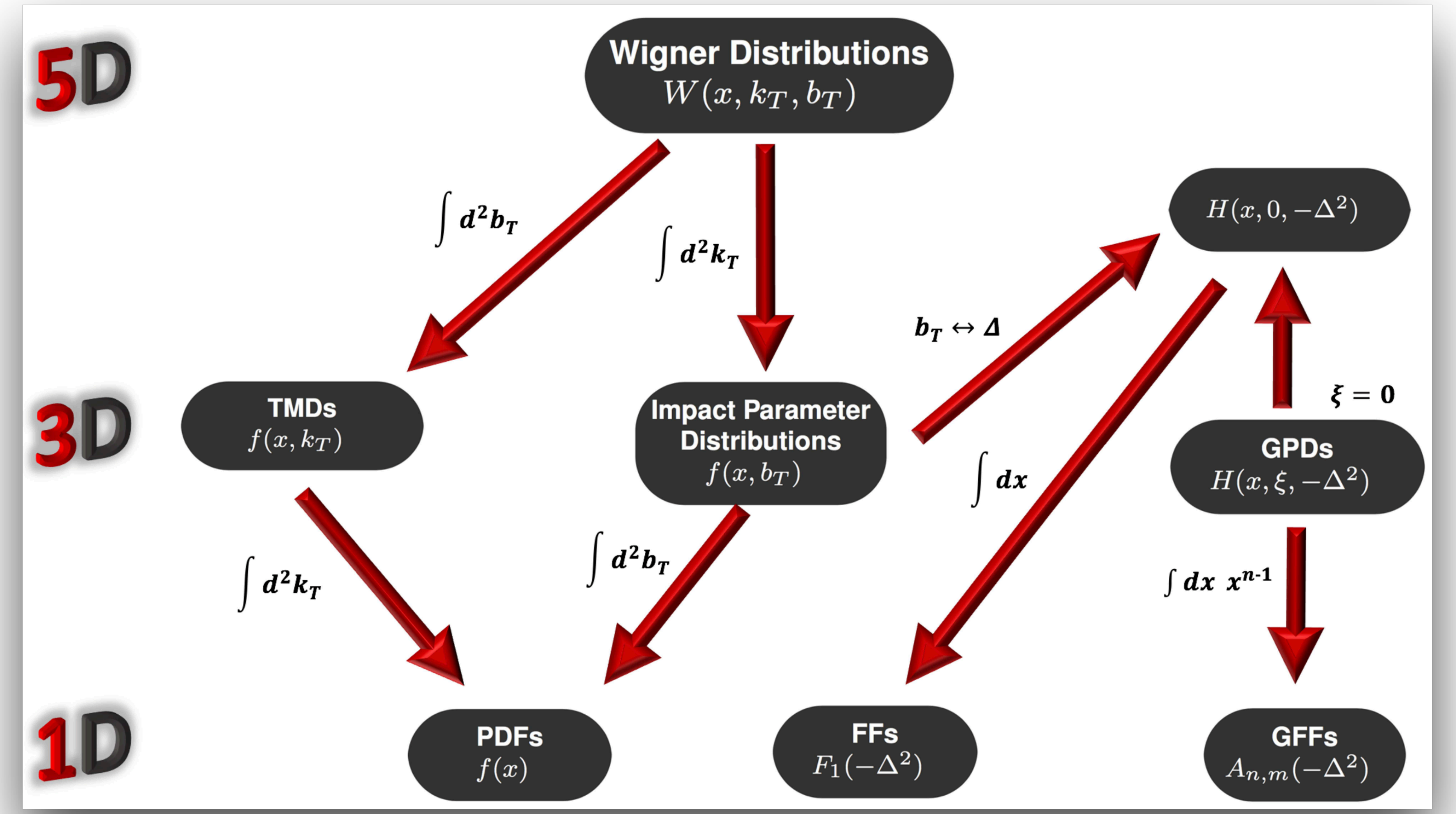

- **Mellin Moments**
  - Embedded information in the PDF
  - As n increases, the larger x region increases in weight

$$\langle x^{n-1} \rangle = \int_{-1}^{1} dx\ x^{n-1} f(x)$$

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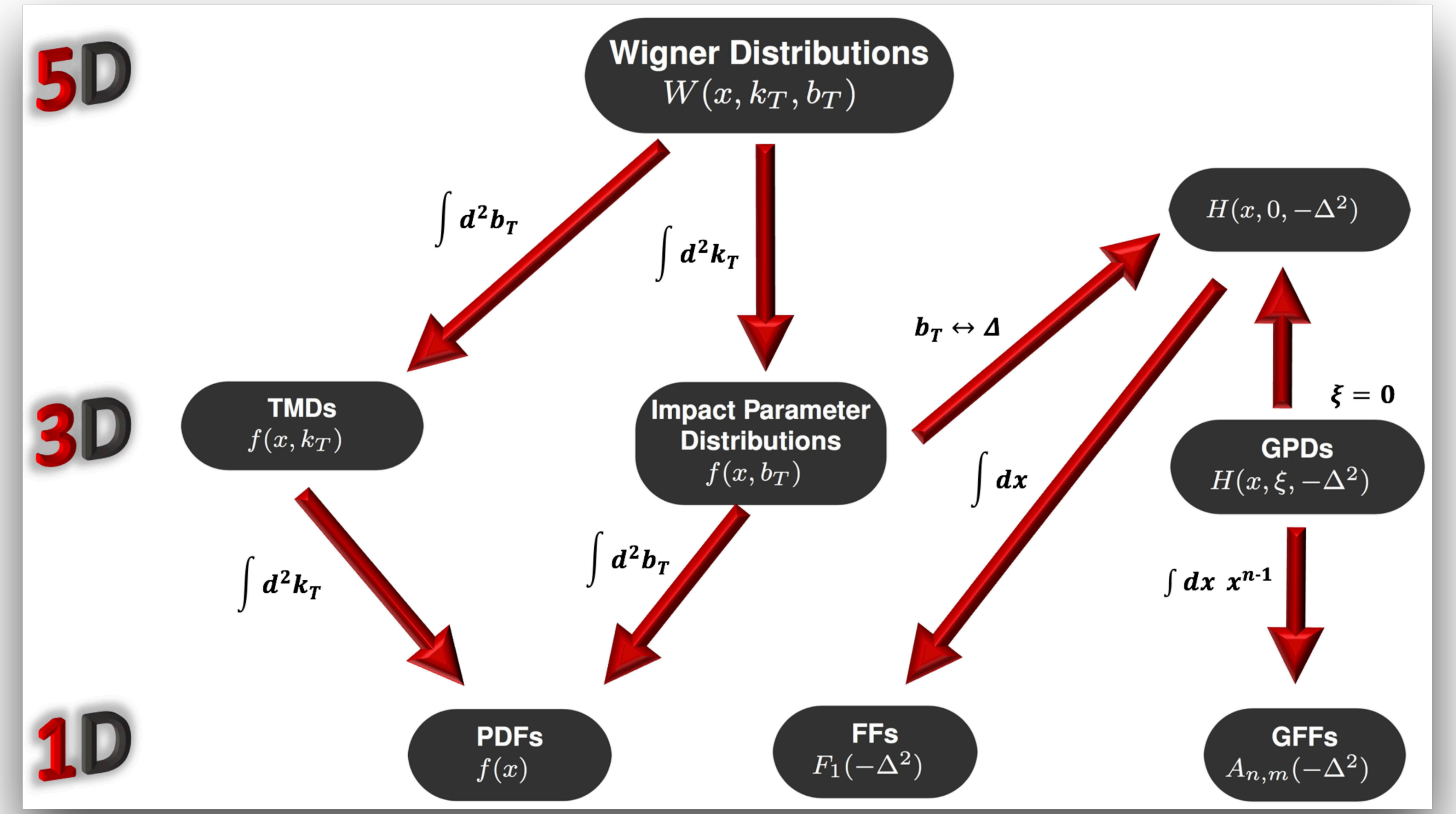

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Multiple methods to calculate moments in Lattice QCD

Mellin Moments from Lattice QCD

- Traditionally, moments are calculated from local operators

$$\partial_\mu \psi = \frac{\psi(x+\hat{\mu}) - \psi(x-\hat{\mu})}{2a}$$

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Locality — Gauge Invariance+Symmetry — Constructed Operator

- Validity comes from Operator Product Expansion (OPE) of a gauge invariant non-local matrix element

$$\langle M(p)\,|\,\bar{\psi}(0)\gamma_\mu \mathscr{W}(0,z)\psi(z)\,|\,M(p)\rangle = \bar{\psi}(0)\gamma_\mu\psi(0) + \bar{\psi}(0)\gamma_\mu D^\nu\psi(0) + \cdots$$

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❖ More derivatives leads to higher order Mellin moments

- ❖ Operator mixing becomes larger at higher moments
- ❖ Moments decrease as n increases
- ❖ $\langle x^3 \rangle$ [Alexandrou, et al. Phys. Rev. D 104, 054504 (2021)] [Alexandrou, et al. arXiv 2605.29998 [hep-lar] (2026)]*
- ❖ $\langle x^5 \rangle$ [Francis, et al. Phys. Rev. D 113, 074520 (2026)] [Francis, et al. Phys. Rev. Lett. 136, 171903 (2026)]

** physical point*

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Difficult to extract higher moments

Mellin Moments from Lattice QCD

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  - Large Momentum Effective Theory (LaMET)
    - Light-cone distribution extracted in $P_3 \to \infty$ limit
      - Matching kernel $\mathcal{O}\left(\frac{\Lambda_{QCD}^2}{x^2 P_3^2}, \frac{\Lambda_{QCD}^2}{(1-x)^2 P_3^2}\right)$
    - $x \in [0.2, 0.8]$ well constrained
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  - Short Distance Expansion → inferred from OPE
    - Reliable for $z \lesssim 0.3$ fm
      - Matching kernel $\mathcal{O}(z^2 \Lambda_{QCD}^2)$
    - Utilize multiple momentum boosts to extract Mellin moments

Lattice Setup

❖ $N_f = 2 + 1 + 1$ Twisted mass fermions with a clover term

Parameters							
Ensemble	β	a [fm]	volume $L^3 \times T$	N_f	m_π [MeV]	Lm_π	L [fm]
cA211.32	1.726	0.093	$32^3 \times 64$	u, d, s, c	260	4	3.0

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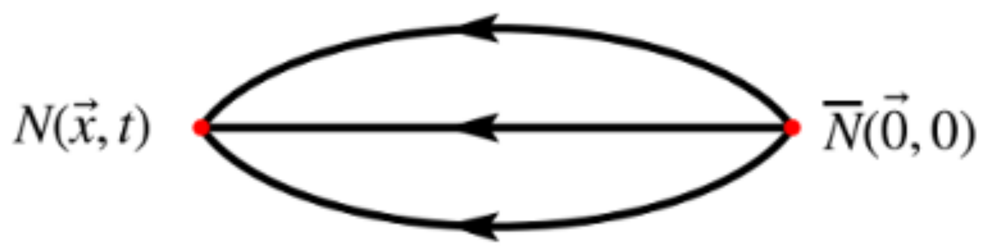
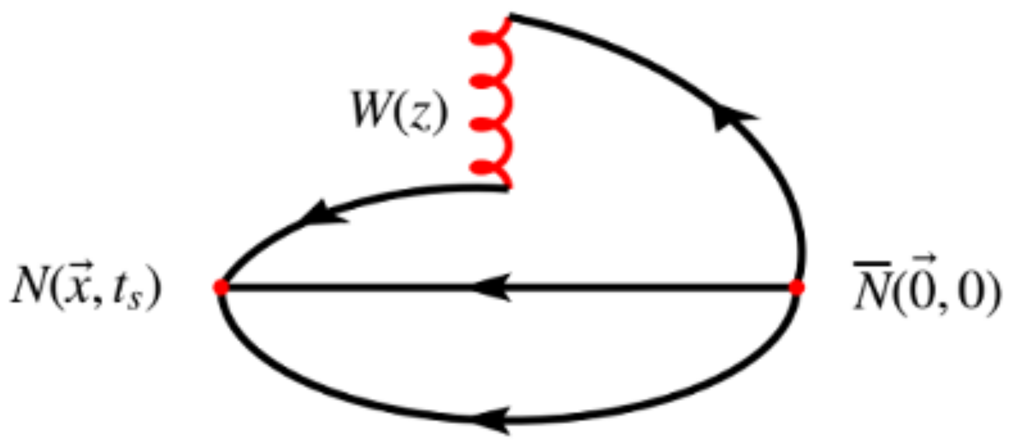
❖ Calculate 2pt and 3pt Correlation functions at both $+P_3$ and $-P_3$

❖ As P_3 increases, higher statistics needed



$\langle N(P) \,|\, N(P) \rangle$



$\langle N(P_f) \,|\, \bar{\Psi}(z)\, \Gamma\, \mathscr{W}(z,0) \Psi(0) \,|\, N(P_i) \rangle$

P_3 [GeV]	0	± 0.41	± 0.83	± 1.25	± 1.66	± 2.07
t_s/a	12	12	12	$10^\star$, $12^\dagger$	10	10
N_{confs}	1,198	1,198	1,198	1,198	1,198	1,198
N^π_{src}	1	8	8	56	84	400
N^K_{src}	1	8	8	8	48	300
N^π_{tot}	1,198	9,584	9,584	67,088	100,632	479,200
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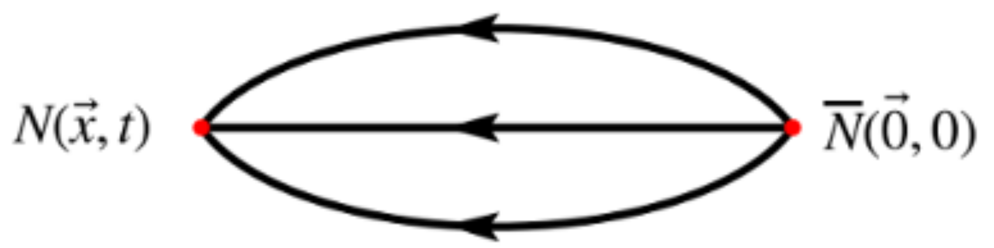

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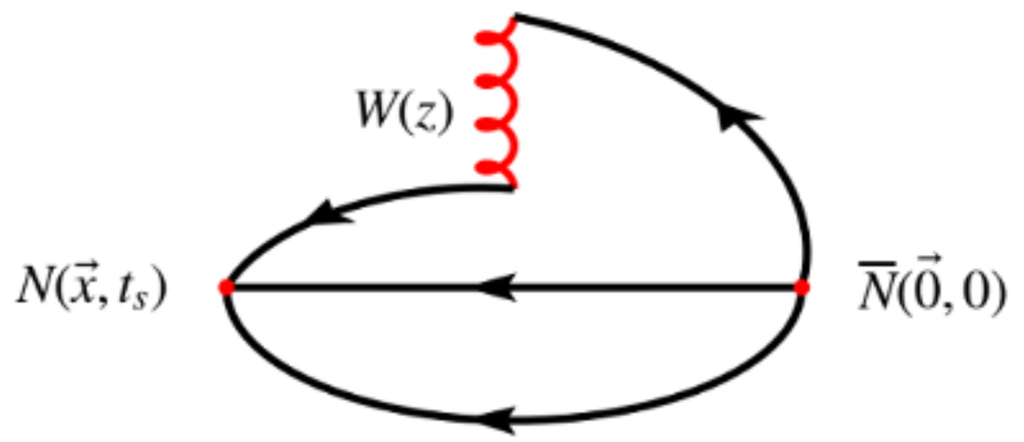

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- Calculate 2pt and 3pt Correlation functions at both $+P_3$ and $-P_3$
  - As P_3 increases, higher statistics needed
- Large t_s needed to ensure ground state is extracted
  - t_s decreased for larger P_3 to ensure good signal to noise ratio
- Kaon requires less statistics since signal is $\sim e^{-m}$

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Lattice Methodology

- ❖ Calculate 2pt and 3pt correlation functions
- ❖ Formulate a ratio

$$R_\mu(z, p; t_s, \tau) = \frac{C_\mu^{3pt}(z, p; t_s, \tau)}{C^{2pt}(p; t_s)}$$

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$\nu = P \cdot z$

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- ❖ Relate renormalized Matrix Elements to Mellin moments by Wilson Coefficients

$$\mathscr{M}(\nu,z^2) = \sum_{n=0}^{\infty} \frac{(i\nu)^n}{n!} c_n(\mu^2 z^2)\langle x^n \rangle$$

$$H_n^{(j)} = \sum_{i=1}^{n} \frac{1}{i^n}$$

At NLO

$$c_n(\mu^2 z^2) = 1 + \frac{\alpha_s C_F}{2\pi}\left[\ln\left(z^2\mu^2 \frac{e^{2\gamma_E+1}}{4}\right)\left(H_n + H_{n+2} - \frac{3}{2}\right) + 2\left(\frac{1}{n+1} - \frac{1}{n+2} - H_n^2 - H_n^{(2)}\right)\right]$$

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- ❖ Reconstruct the valence distribution

$$\langle x^n \rangle = \int_0^1 \frac{x^{n+\alpha}(1-x)^\beta}{B(\alpha+1,\beta+1)}$$

Lattice Methodology

Fixed z Analysis

$$\chi^2 = \sum_{P_3} \frac{(\mathrm{Re/Im}[\mathscr{M}^{SDF}(\nu, z^2)] - \mathrm{Re/Im}[\mathscr{M}^{data}(\nu, z^2)])^2}{\sigma^2_{\mathrm{Re/Im}}}$$

- ❖ Fit even moments (real part) and odd moments (imaginary part) separately
- ❖ Test the proper truncation for the real and imaginary part separately
- ❖ Allows one to see residual $z-$ dependence

Lattice Methodology

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Combined P_3 and z Analysis

$$\chi^2 = \sum_{P_3} \sum_{z=z_{min}}^{z_{max}} \left(\frac{(\mathrm{Re}[\mathscr{M}^{SDF}(\nu, z^2)] - \mathrm{Re}[\mathscr{M}^{data}(\nu, z^2)])^2}{\sigma^2_{\mathrm{Re}}} + \frac{(\mathrm{Im}[\mathscr{M}^{SDF}(\nu, z^2)] - \mathrm{Im}[\mathscr{M}^{data}(\nu, z^2)])^2}{\sigma^2_{\mathrm{Im}}} \right)$$

- Fit all moments at once
  - P_3 and z separated since combinations that lead to the same Ioffe value are not statistically equivalent
- Test different ranges for z_{min} and z_{max}
- Allows one to see systematic effects of higher twist contamination

Lattice Methodology

Fixed z Analysis

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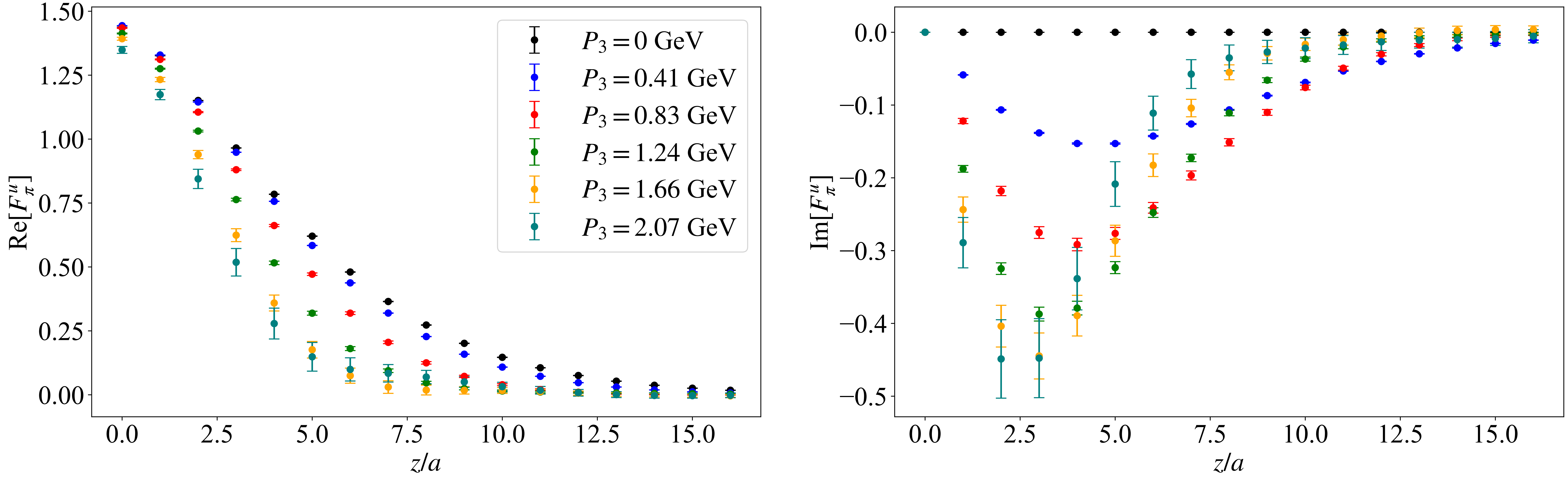
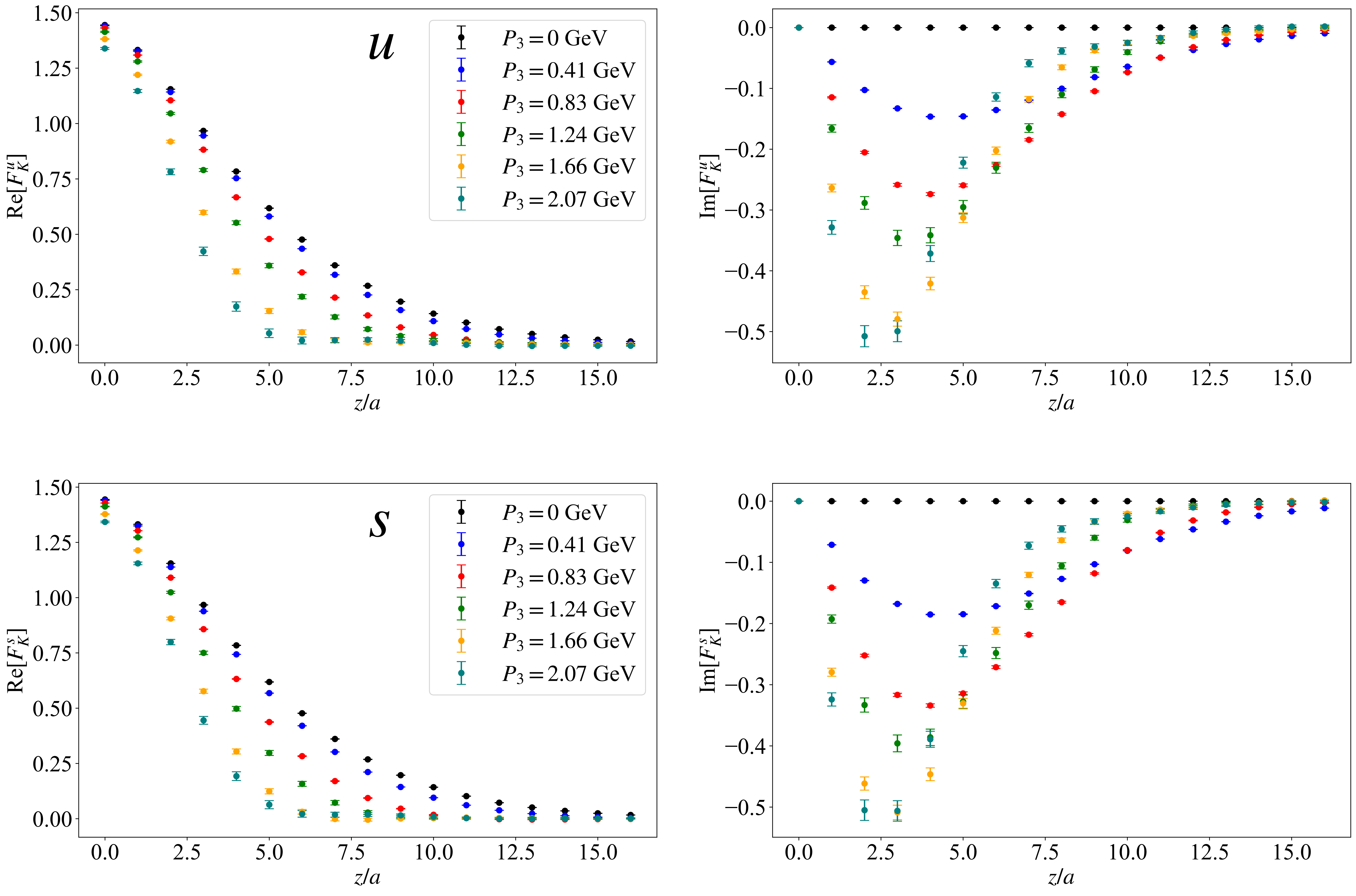
Majority of results are from using fixed $\mu = 2$ GeV

Matrix Elements: Pion



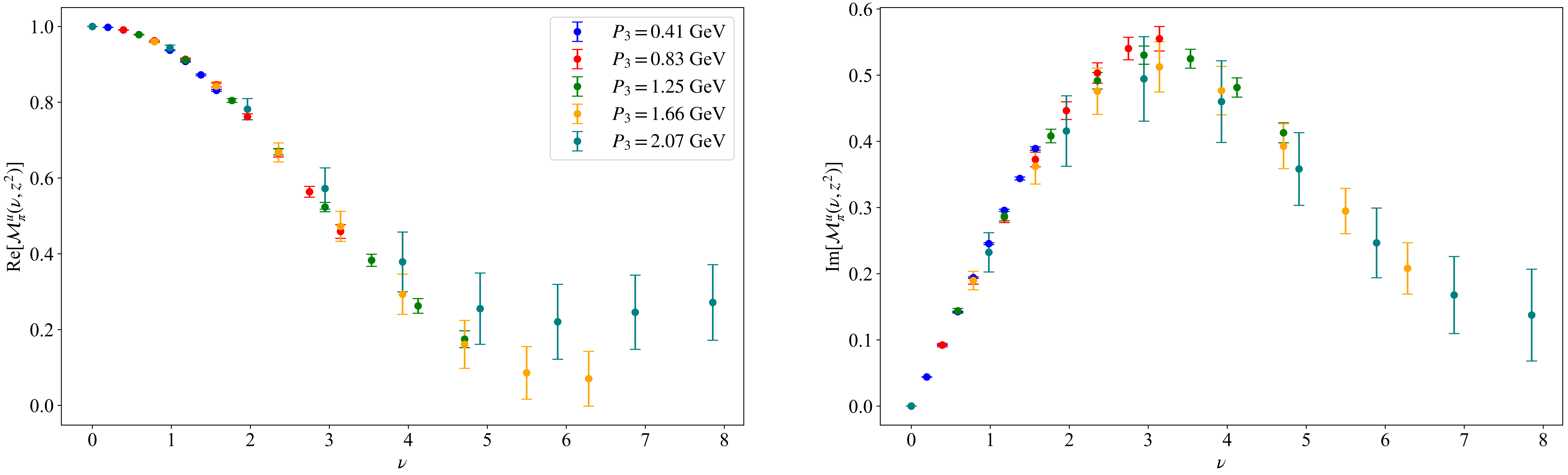
- ❖ Good signal to noise ratio
  - ❖ Increased statistics for larger P_3 still present larger error
- ❖ Real part becomes more narrow as momentum increases
- ❖ Imaginary part has a larger peak as momentum increases
- ❖ Symmetrized by $F_M^f(-z \cdot P) = F_M^{f*}(z \cdot P)$

Matrix Elements: Kaon



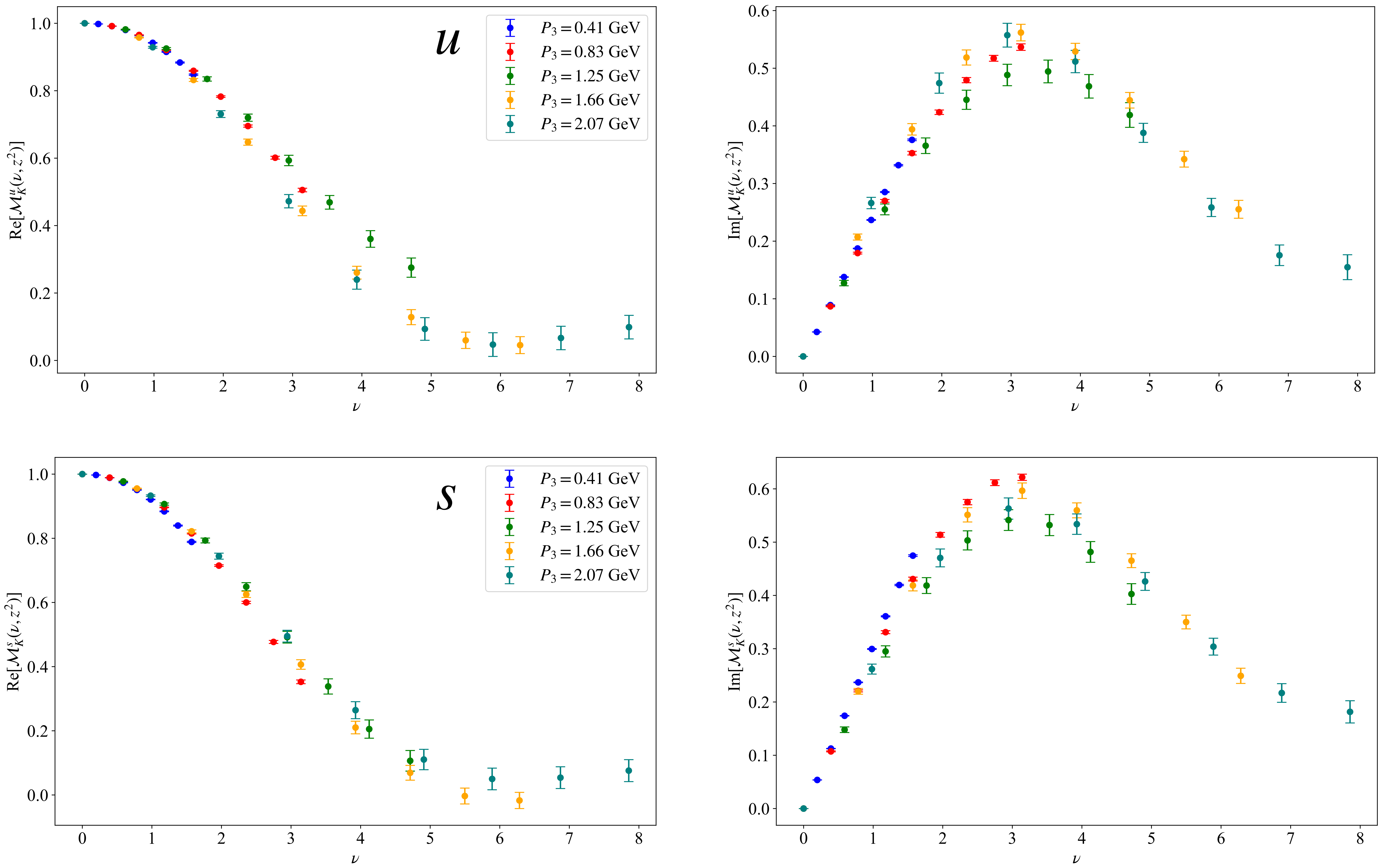
- ❖ Calculation of the up-quark and strange-quark separately
- ❖ Good signal to noise ratio
  - ❖ Increased statistics for larger P_3 still present larger error
- ❖ Similar trends as the pion
- ❖ More accurate than the pion
- ❖ Symmetrized by $F_M^f(-z \cdot P) = F_M^{f*}(z \cdot P)$

Double Ratio: Pion



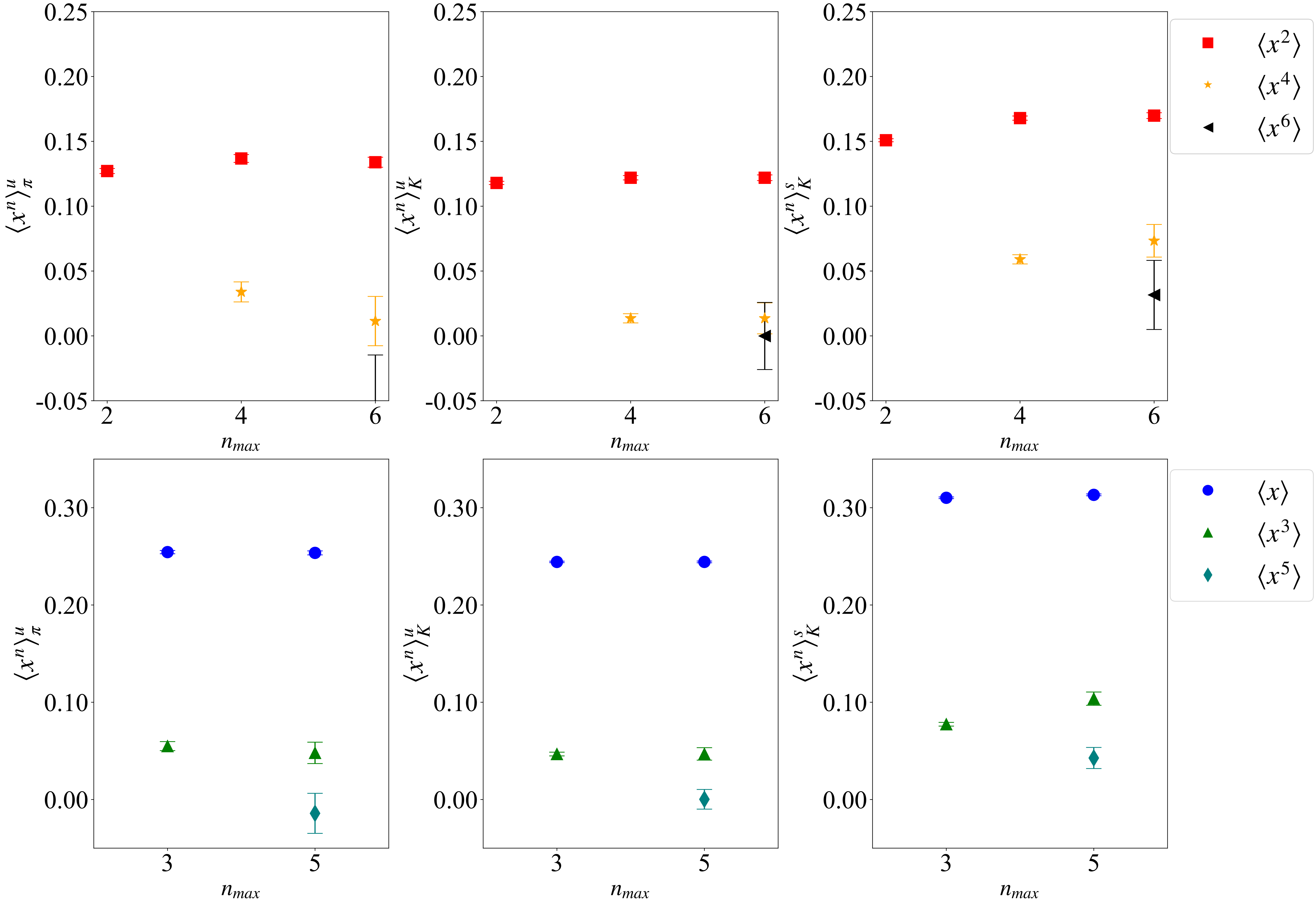
- Good signal to noise ratio
  - Large ν noisier and deviates between $P_3 = 1.66$ and 2.07 GeV
    - However this out outside the acceptable z region
- Real part decays towards zero
- Real part at $\nu = 0$ expected to be 1 since $\langle x^0 \rangle = \int_{-1}^{1} dx\ q(x) = 1$

Double Ratio: Kaon



- ❖ Better signal to noise ratio than the pion
- ❖ Similar trends as the pion

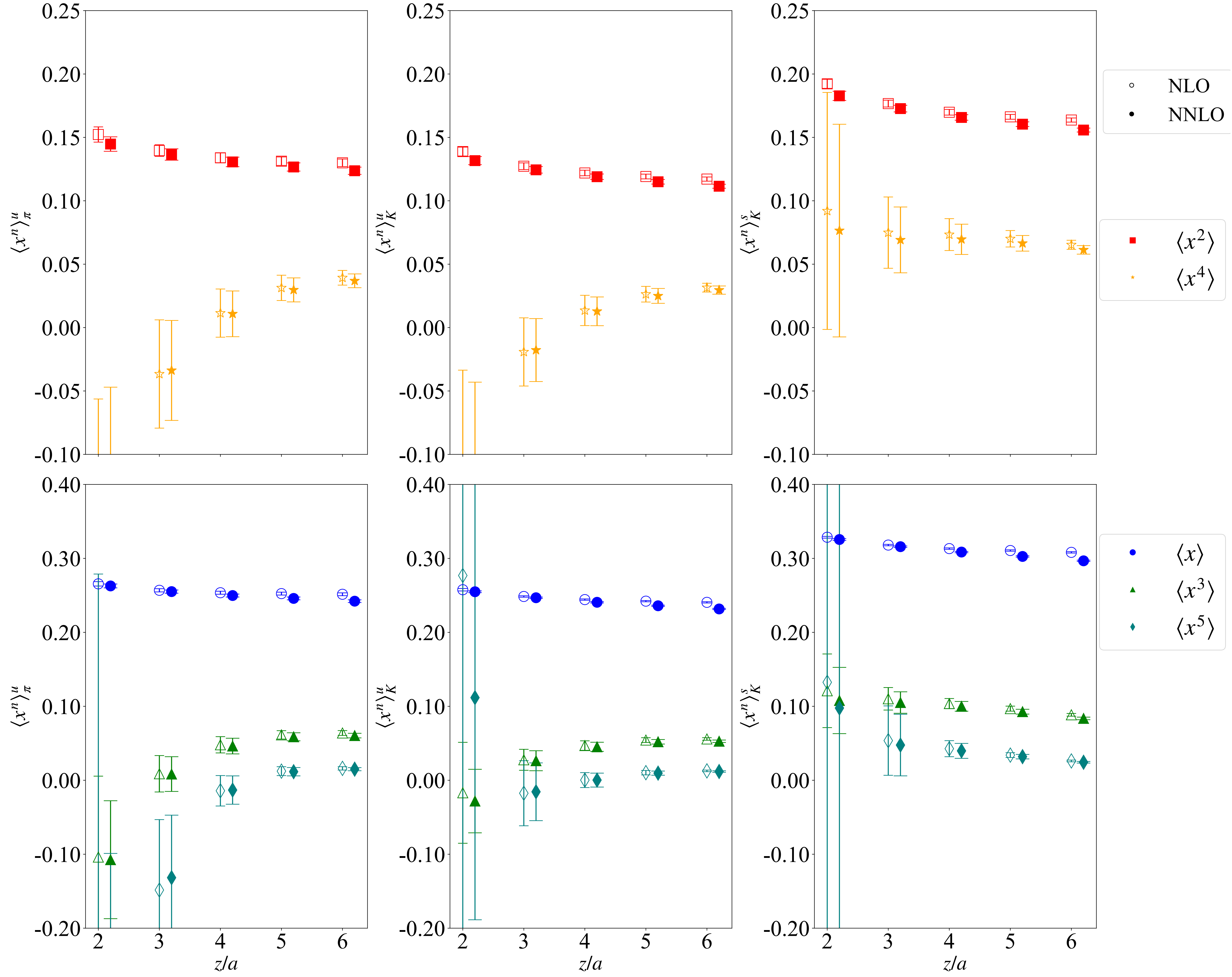
Fixed z Analysis: n_{max} Test



$$\mathscr{M}(\nu, z^2) = \sum_{n=0}^{\infty} \frac{(i\nu)^n}{n!} C_n(\mu^2 z^2) \langle x^n \rangle$$

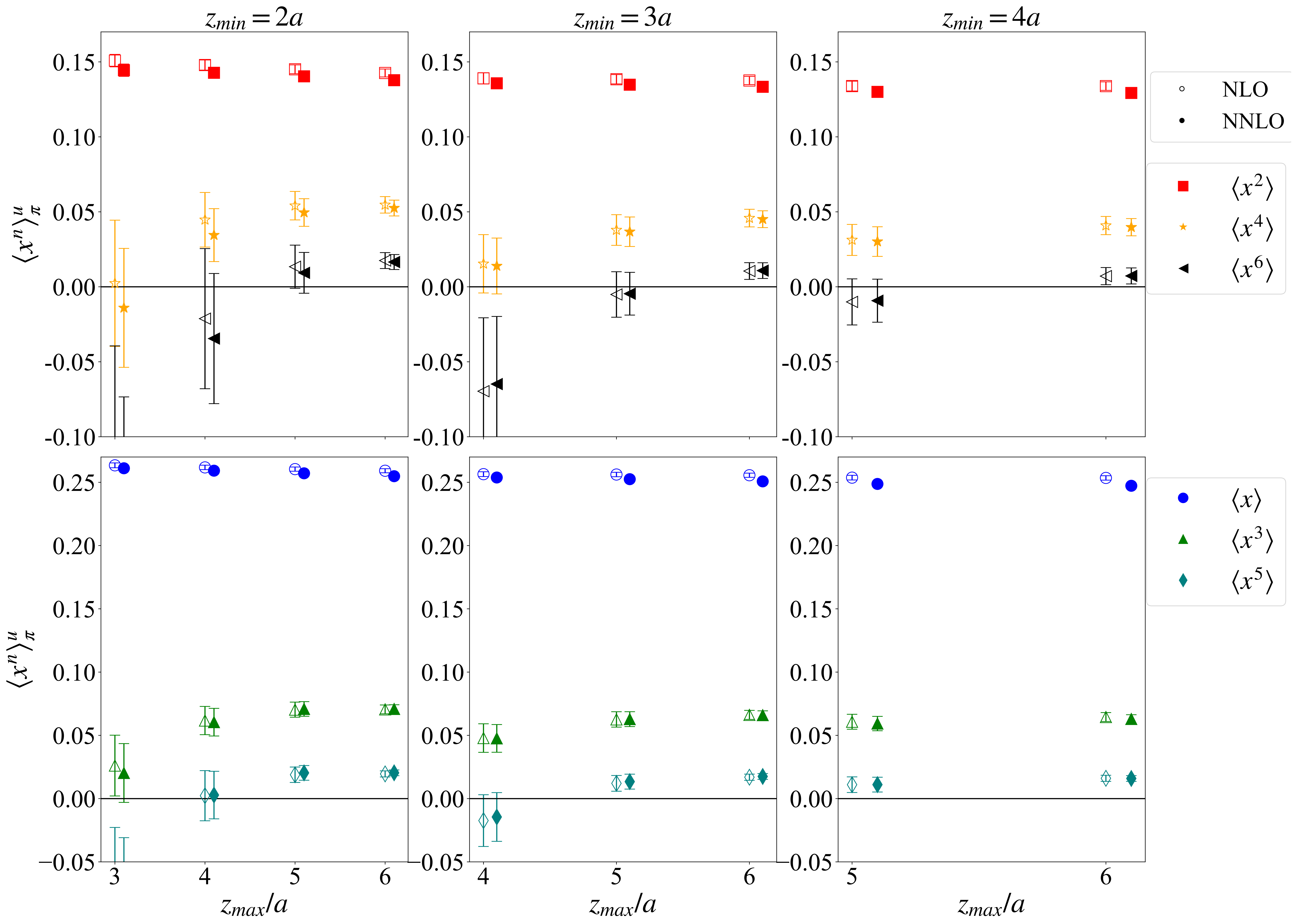
- ❖ Real and imaginary parts fit separately
  - ❖ Real part $\rightarrow \langle x^{2m} \rangle$
  - ❖ Imaginary part $\rightarrow \langle x^{2m+1} \rangle$
- ❖ Stability as n_{max} increases
  - ❖ $n_{max} = 6$ for the real part
  - ❖ $n_{max} = 5$ for the imaginary part

Fixed z Analysis: NLO vs NNLO C_n



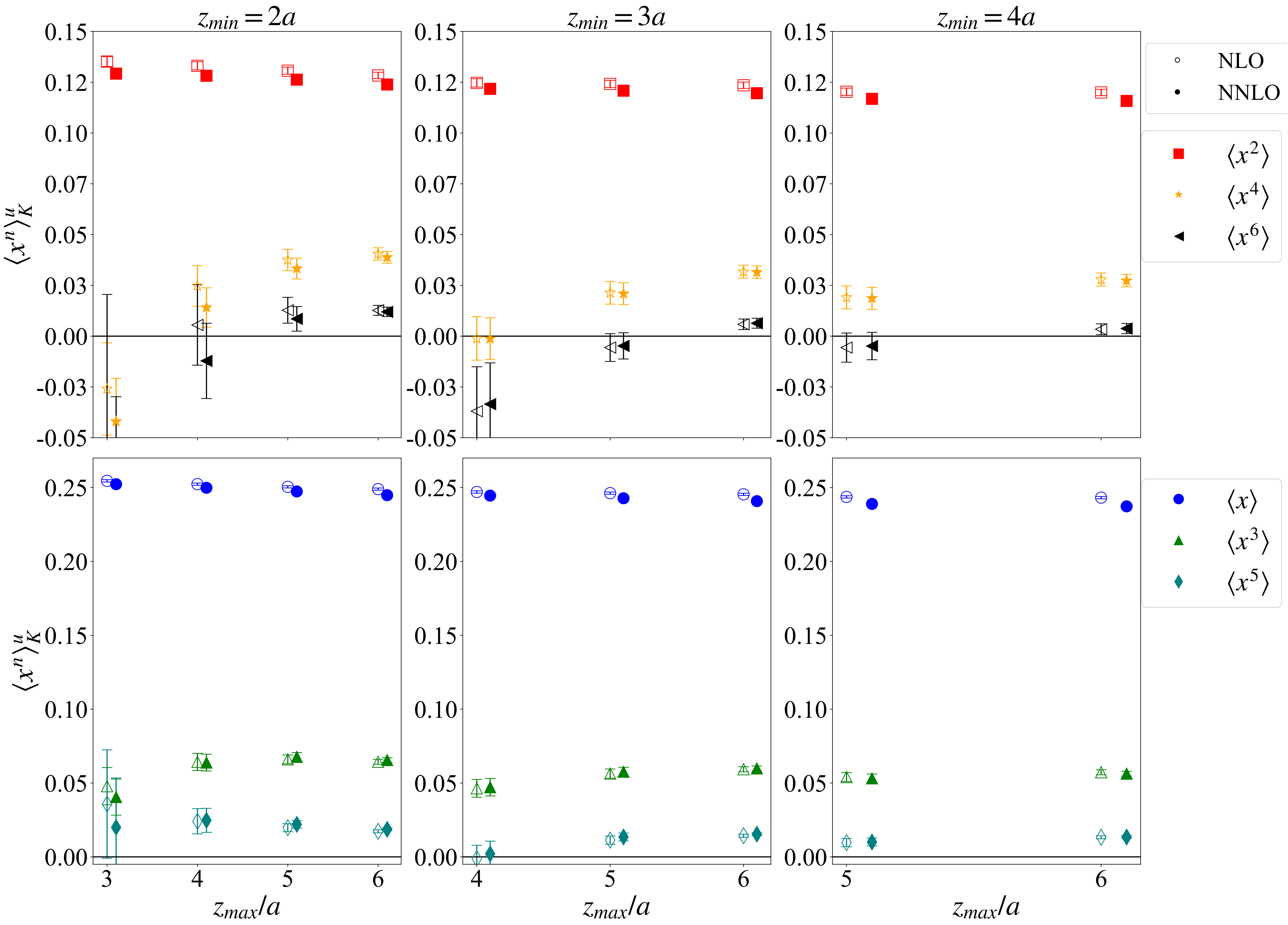
- ❖ Moments extracted from NLO and NNLO Wilson coefficients have a minimal shift
- ❖ Nontrivial $z-$dependence of the moments
- ❖ Higher moments stabilize with larger z
- ❖ No plateau until large z values
  - ❖ Indication we need to implement combined P_3 and z fits

Combined P_3 and z Analysis: $z-$ranges

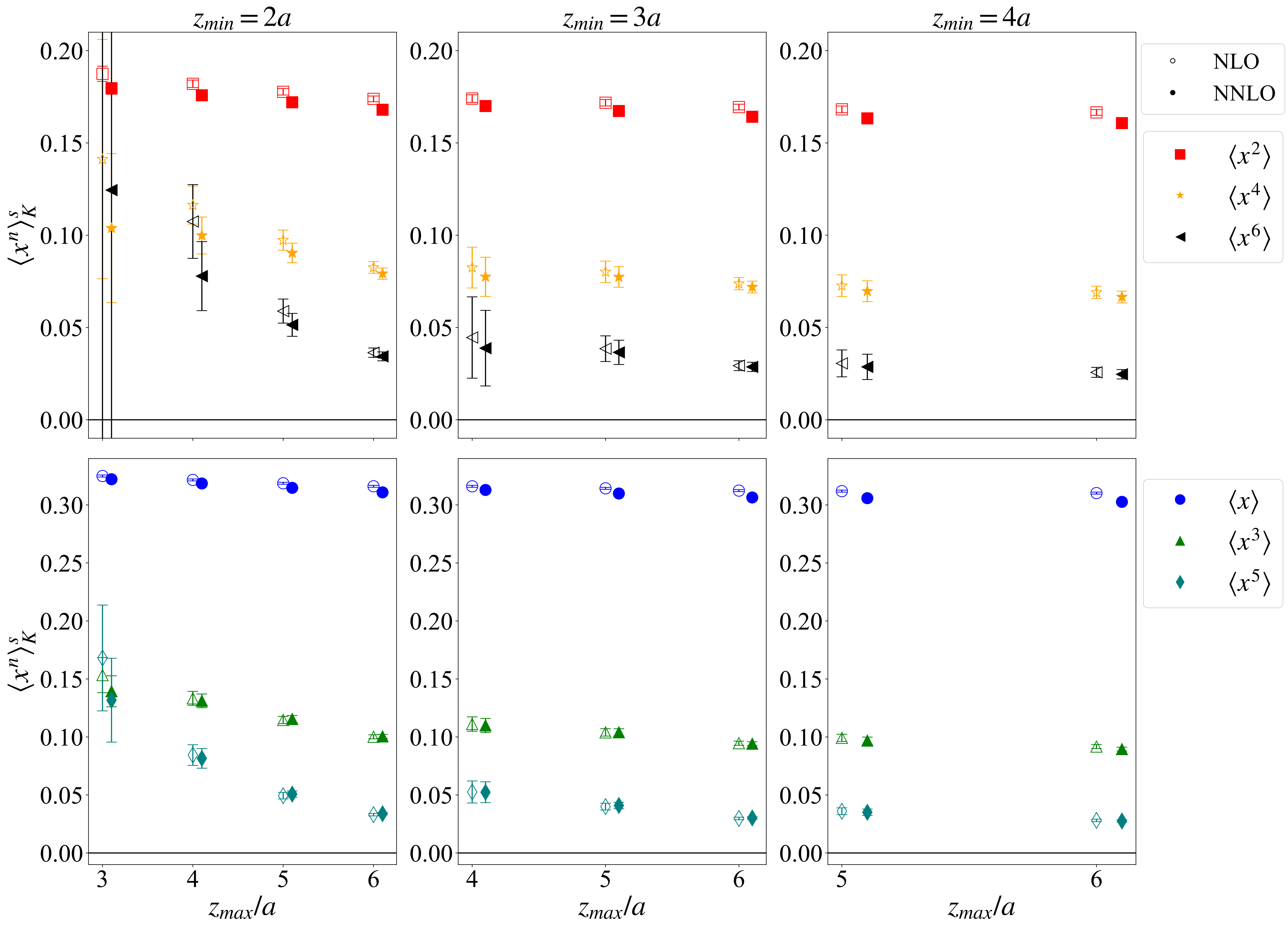


- Moments extracted from NLO and NNLO Wilson coefficients have a minimal shift
- Avoid $z_{min} = a$ due to discretization effects
- Inclusion of higher z allows for higher moments
- Systematics in different choices of z_{min} and z_{max}
- Final results are a weighted average of $z_{\rm min} \in [3a, 4a]$ and $z_{max} \in [4a, 6a]$ with $z_{max} - z_{min} \geq a$

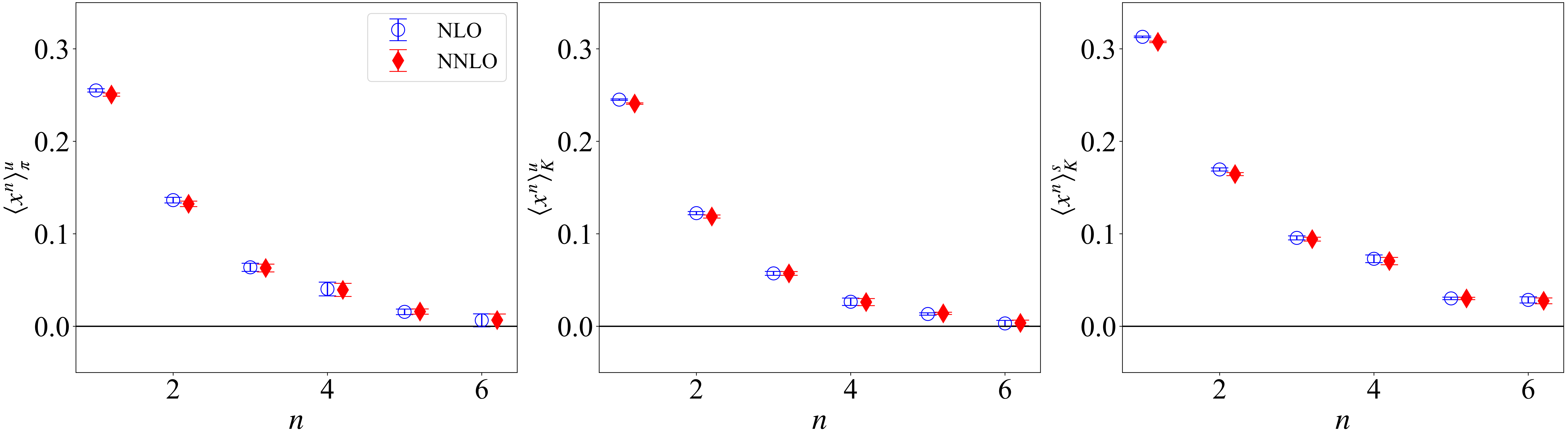
Combined P_3 and z Analysis: z— ranges



Combined P_3 and z Analysis: $z-$ ranges



Combined P_3 and z Analysis: Final Results



- Moments at both NLO and NNLO accuracy have a small difference between results
- PDFs decay to zero as $x \to 1$
  - As n increases, larger weight is given to higher x, causing the moment to be close to zero
- The strange-quark in the kaon has larger moments at each n
  - This indicates the strange quark carries more of the hadron momentum than the up-quark

DGLAP Evolution

- ❖ Unlike local operators, we have perturbative dependent results
  - ❖ C_n calculated at NLO, NNLO, …

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- C_n now becomes:

$$C_n^{NLOevo}(\mu^2, z^2) = C_n^{NLO}(\mu_0^2, z^2)\left(\frac{\alpha_s(\mu_0)}{\alpha_s(\mu)}\right)^{\gamma_n^{(0)}/\beta_0}$$

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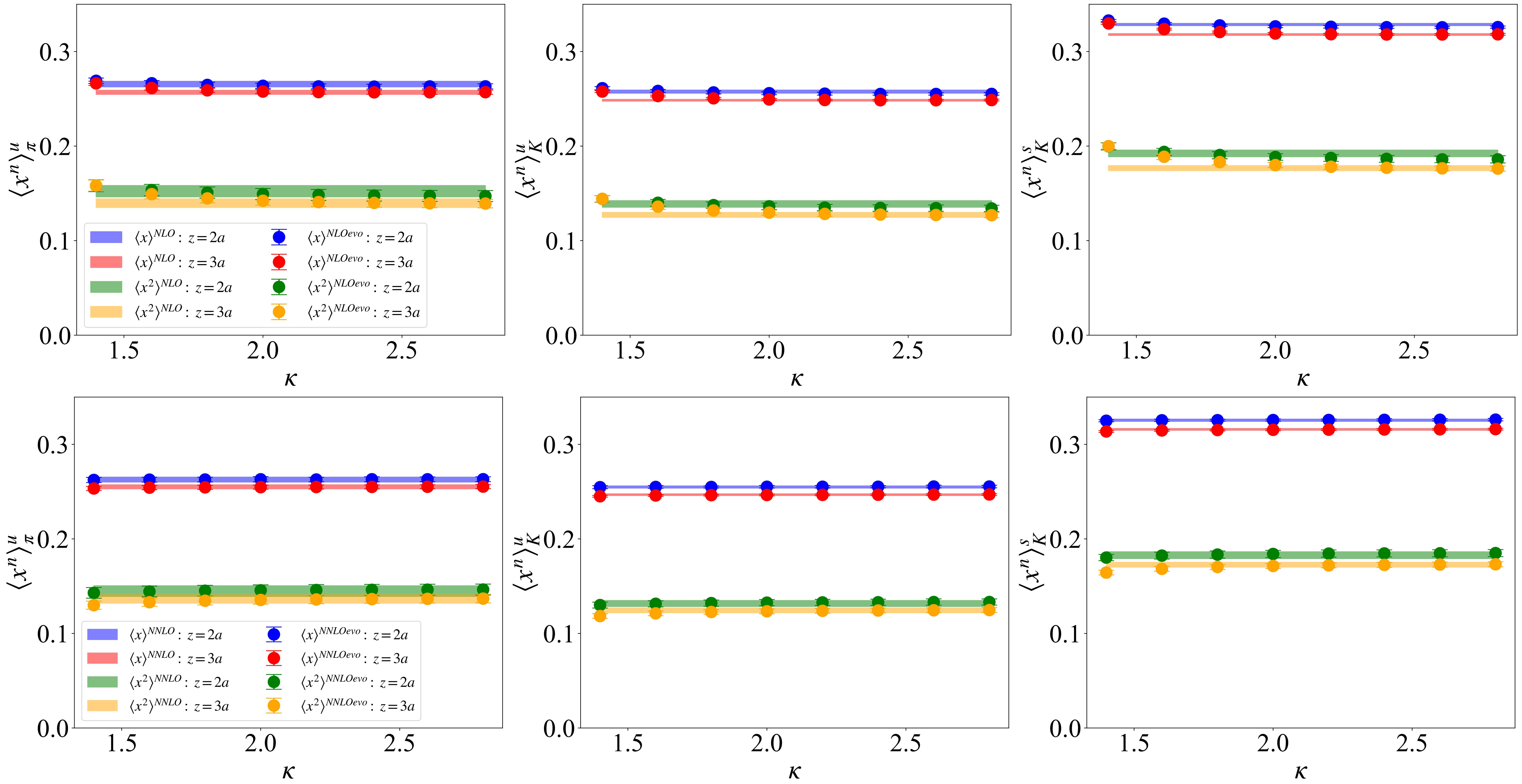
$$C_n^{NNLOevo}(\mu^2, z^2) = C_n^{NNLO}(\mu_0^2, z^2)e^{\frac{-\gamma_n^{(0)}}{\beta_0}\ln\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)-\left(\frac{\gamma_n^{(1)}}{\beta_1}-\frac{\gamma_n^{(0)}}{\beta_0}\right)\ln\left(\frac{4\pi\beta_0+\beta_1\alpha_s(\mu)}{4\pi\beta_0+\beta_1\alpha_s(\mu_0)}\right)}$$

DGLAP Evolution



- Less dependence on κ for NNLOevo
  - $\langle x^2 \rangle$ requires larger κ to agree with the fixed μ results at NLOevo
- NNLOevo consistent with fixed μ results

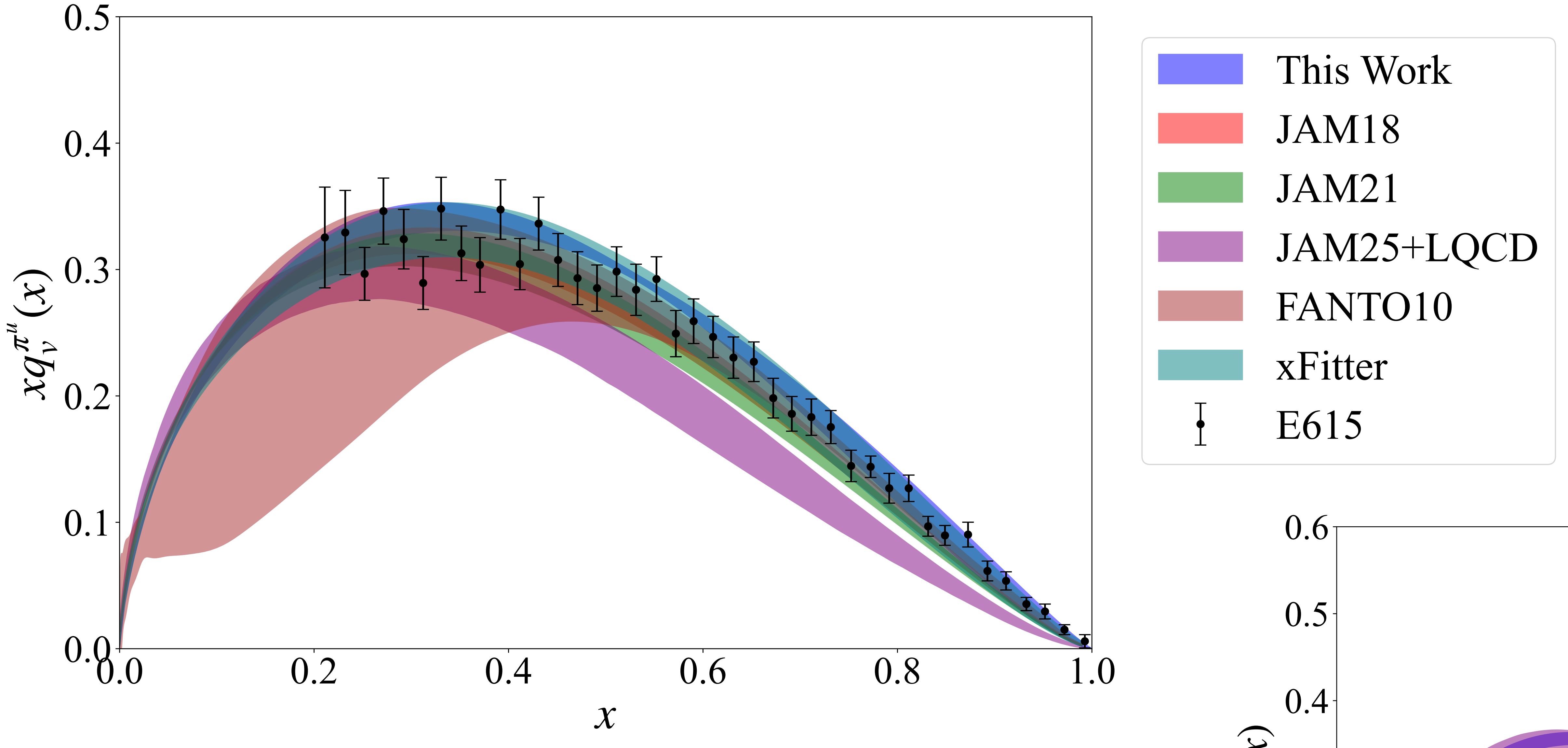
Results Comparison

- ❖ Good agreement with other calculation
  - ❖ Not a 1:1 comparison due to different pion masses, lattice spacing, etc.

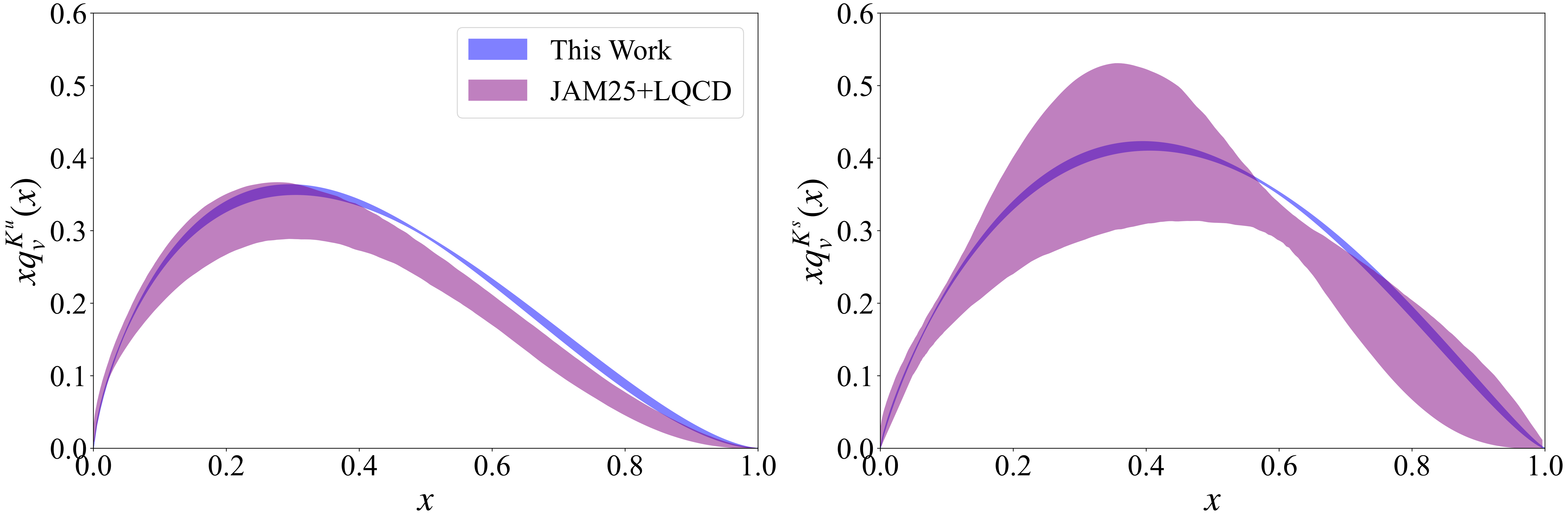
	$\langle x \rangle$		$\langle x^2 \rangle$		$\langle x^3 \rangle$	
	Nonlocal	Local [29]	Nonlocal	Local [77]	Nonlocal	Local [77]
π^u	0.251(5)	0.273(9)	0.132(6)	0.110(6)	0.063(10)	0.026(17)
K^u	0.241(6)	0.262(3)	0.119(5)	0.101(2)	0.057(6)	0.043(7)
K^s	0.308(7)	0.332(3)	0.165(7)	0.145(2)	0.094(10)	0.079(6)

Results Comparison

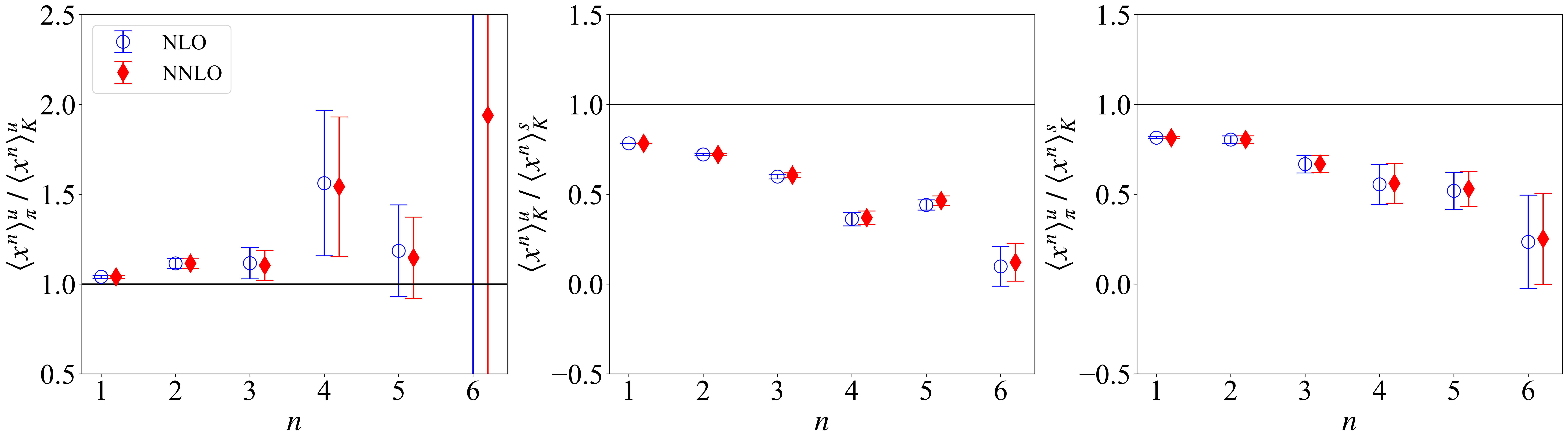
- Odd n give moments related to q_{v2s}
  - $\langle x^{2m+1} \rangle \approx \langle x^{2m+1} \rangle_v$
    - Focusing only on connected diagrams, treat sea contribution as negligible



$\mu = 5.2$ GeV



SU(3) Symmetry Breaking



- ❖ Pion and kaon up-quark are similar with statistical fluctuations
  - ❖ Up to 4% difference from unity
- ❖ Flavor dependence much stronger in comparison of up-quark and strange-quark
  - ❖ Deviation from unity becomes stronger as n increases, but lower moments give a more robust conclusion
- ❖ Consistent with previous studies

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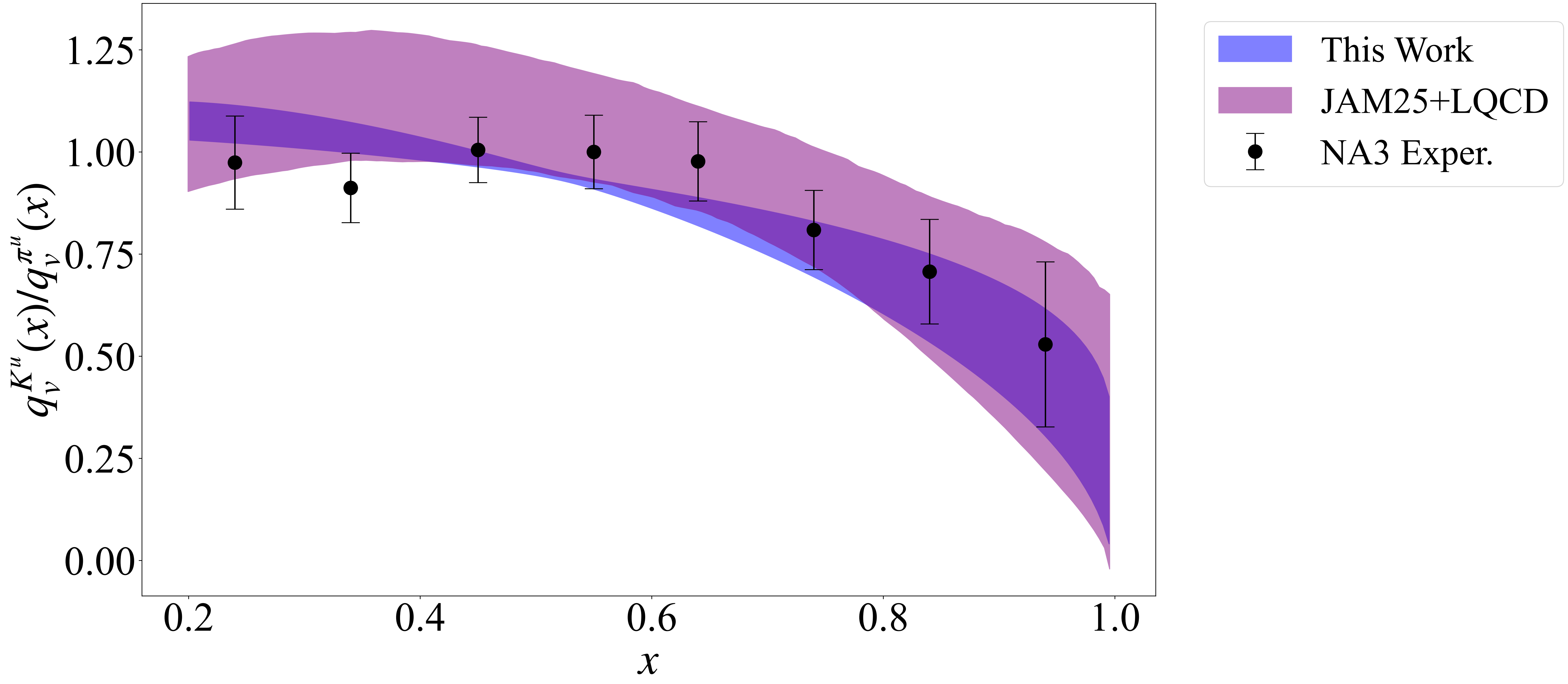

Summary and Future Work

- Calculate Mellin moments at NLO and NNLO accuracy
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  - Extract up to $\langle x^6 \rangle$
  - Reconstruct valence distribution
- Test perturbative accuracy through DGLAP evolution at NLO and NNLO accuracy
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- Test perturbative accuracy through DGLAP evolution at NLO and NNLO accuracy
- Test SU(3) symmetry breaking
- Continue work to Generalized Form Factors
- Take continuum limit
- Implement Neural Networks

Thank You!!!

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- Interdisciplinary Centre for Mathematical and Computational Modeling of the Warsaw University by Okeanos
- Academic Computer Center in Gdańsk by Tryton

Award Number:
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DOE Early Career Award (NP)
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Backup Slides

Wilson Coefficient Derivation

❖ Focusing only on NLO accuracy, start with known matching relation

$$\mathscr{M}(\nu, z^2) = \mathscr{Q}(\nu, \mu^2) + \frac{\alpha_s C_F}{2\pi} \int_0^1 du \left[\ln\left(z^2 \mu^2 \frac{e^{2\gamma_E + 1}}{4} \right) \left(\frac{1+u^2}{u-1} \right)_+ + \left(4 \frac{\ln(1-u)}{u-1} - 2(u-1) \right)_+ \right] \mathscr{Q}(u\nu, \mu^2)$$

$$\int dx \, [f(x)]_+ g(x) = \int dx \, f(x)(g(x) - g(1))$$

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❖ Note that $\mathscr{Q}$ is the Fourier transform on the light-cone PDF

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❖ Substitute in and apply plus prescription

$$\mathcal{M}(\nu, z^2) = \sum_{n=0}^{\infty}\frac{(i\nu)^n}{n!}\langle x^n\rangle\left[1 + \frac{\alpha_s C_F}{2\pi}\int_0^1 du \left[\ln\left(z^2\mu^2\frac{e^{2\gamma_E+1}}{4}\right)\left(\frac{1+u^2}{u-1}\right) + \left(4\frac{\ln(1-u)}{u-1} - 2(u-1)\right)\right](u^n - 1)\right]$$

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- Solve the integrals

$$\mathcal{M}(\nu, z^2) = \sum_{n=0}^{\infty} \frac{(i\nu)^n}{n!} \langle x^n \rangle \left[1 + \frac{\alpha_s C_F}{2\pi} \left[\ln\left(z^2 \mu^2 \frac{e^{2\gamma_E+1}}{4} \right) \left(H_n + H_{n+2} - \frac{3}{2} \right) + 2\left(\frac{1}{n+1} - \frac{1}{n+2} - H_n^2 - H_n^{(2)} \right) \right] \right]$$

DGLAP Evolution Derivation

❖ Start with the evolution equation for Wilson coefficients

$$\ln\left(\frac{C_n(\mu)}{C_n(\mu_0)}\right) = \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} d(\alpha_s(\mu'))\,\frac{\gamma_n(\alpha_s(\mu'))}{\beta(\alpha_s(\mu'))}$$

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❖ Substitute the order you want into the integral and solve

❖ NLOevo

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❖ NNLOevo

$$\ln\left(\frac{C_n(\mu)}{C_n(\mu_0)}\right) = \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} d(\alpha_s(\mu'))\,\frac{\frac{\alpha_s}{4\pi}\gamma_n^{(0)} + \frac{\alpha_s^2}{(4\pi)^2}\gamma_n^{(1)}}{-\frac{\alpha_s^2}{4\pi}\beta_0 - \frac{\alpha_s^3}{(4\pi)^2}\beta_1} = -\frac{\gamma_n^{(0)}}{\beta_0}\ln\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right) - \left(\frac{\gamma_n^{(1)}}{\beta_1} - \frac{\gamma_n^{(0)}}{\beta_0}\right)\ln\left(\frac{4\pi\beta_0 + \beta_1\alpha_s(\mu)}{4\pi\beta_0 + \beta_1\alpha_s(\mu_0)}\right)$$