The Intrinsic Soft Function in the Coulomb-Gauge Approach

The XIIIth Meeting on Lattice Parton Physics from
Large Momentum Effective Theory (LaMET 2026)
Jagiellonian University, Krakow, Poland
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Y. Zhao, Phys.Rev.Lett. 133 (2024), 241904.
D. Bollweg, X. Gao, J. He, S. Mukherjee, and Y. Zhao, Phys.Rev.D 112 (2025) 3, 034501.

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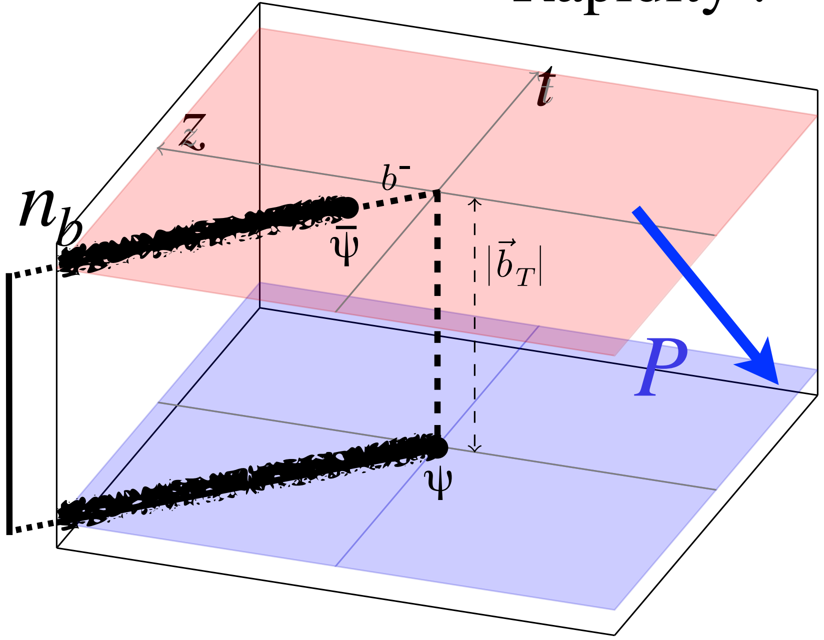
TMD definition in QCD

- Beam function:

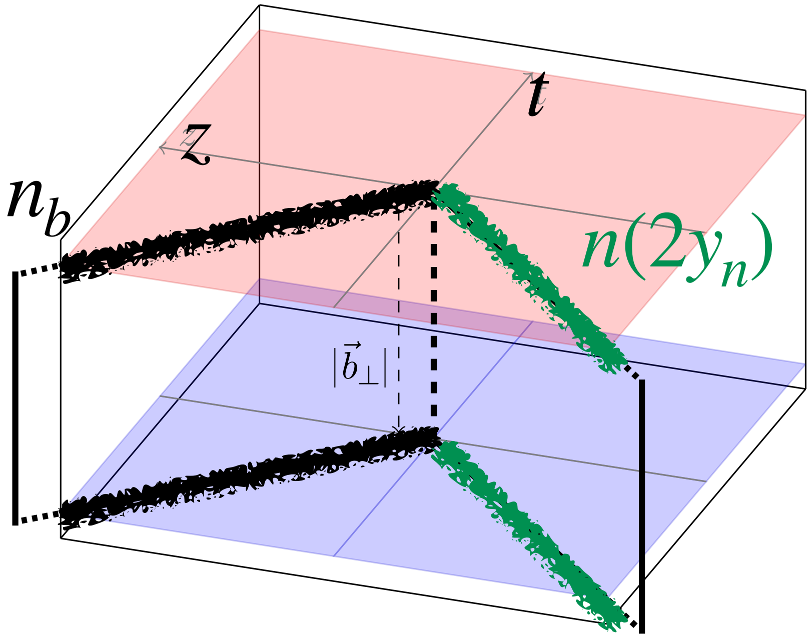
$$n_b^2 = 0$$

$$\text{Rapidity}: \quad y_B = \frac{1}{2}\ln\left|\frac{n_b^+}{n_b^-}\right| = -\infty$$



Hadronic matrix element

- Soft function :



Vacuum matrix element

$$f_i(x, \mathbf{b}_T, \mu, \zeta) = \lim_{\epsilon\to 0} Z_{\rm UV}(\epsilon, \mu, \zeta) \lim_{y_B\to-\infty} \frac{B_i(x, b_T, \epsilon, y_P - y_B)}{\sqrt{S^q(b_T, \epsilon, 2(y_n - y_B))}}$$

Collins-Soper scale: $\zeta = 2(xP^+e^{-y_n})^2 = 4x^2m_N^2e^{2(y_P-y_n)}$

Rapidity divergence regulator

TMD definition in QCD

- Beam function:
- Soft function :

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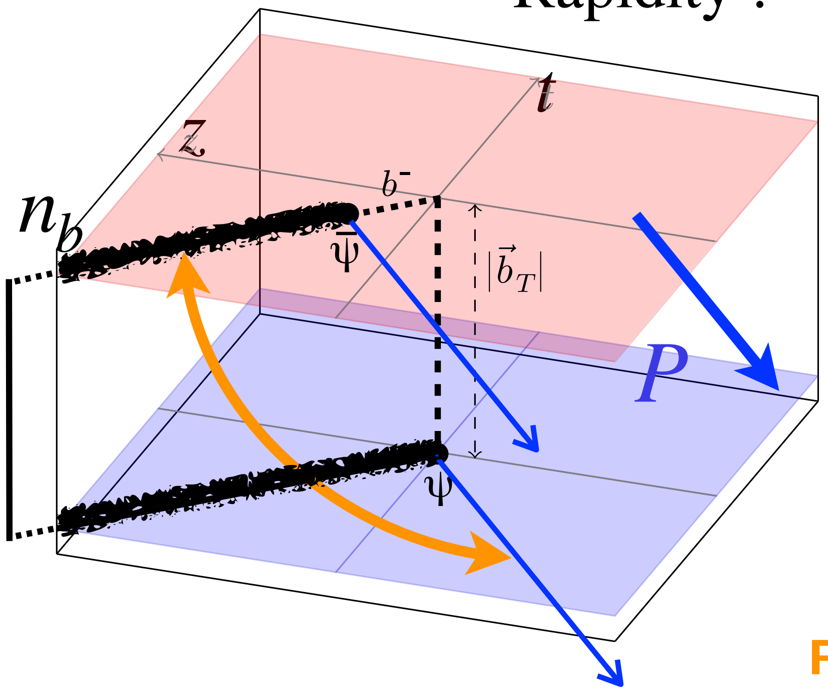
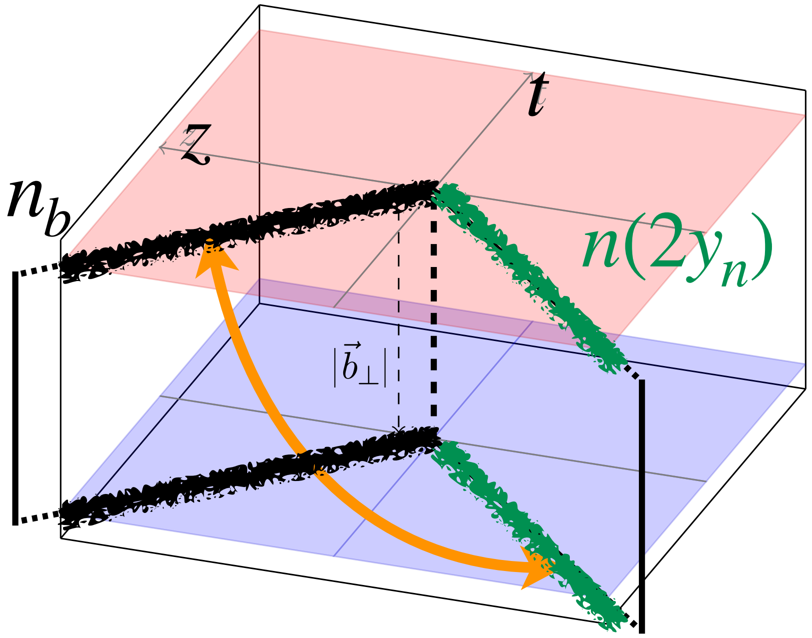


Hadronic matrix element

Rapidity divergences



Vacuum matrix element

$$f_i(x, \mathbf{b}_T, \mu, \zeta) = \lim_{\epsilon\to 0} Z_{\mathrm{UV}}(\epsilon, \mu, \zeta) \lim_{y_B\to-\infty} \frac{B_i(x, b_T, \epsilon, y_P - y_B)}{\sqrt{S^q(b_T, \epsilon, 2(y_n - y_B))}}$$

Collins-Soper scale: $\zeta = 2(xP^+e^{-y_n})^2 = 4x^2m_N^2e^{2(y_P-y_n)}$

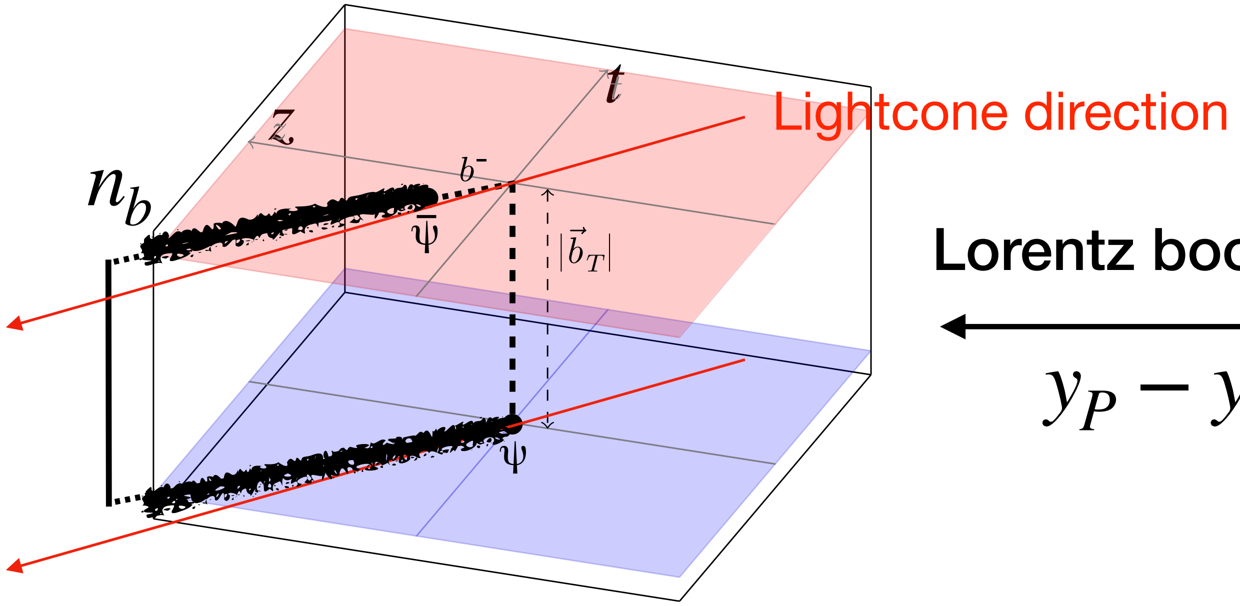
Rapidity divergence regulator

TMDs from LaMET

- Beam function (in Collins' scheme):

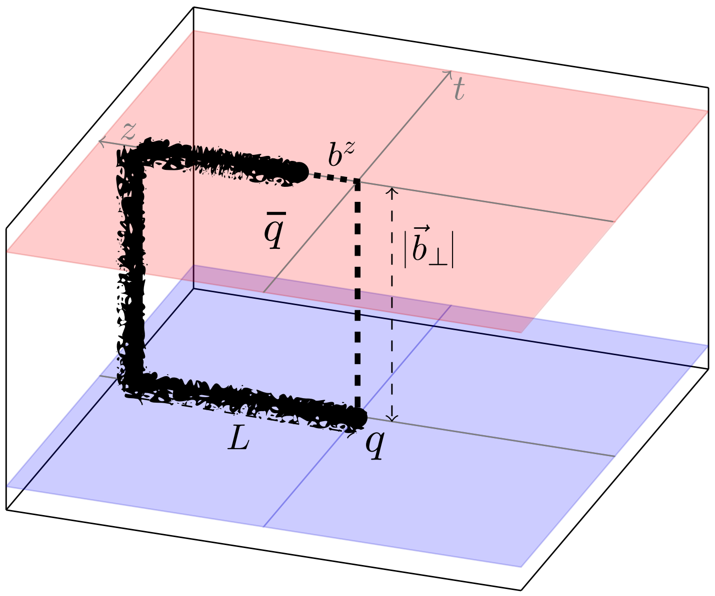
Collins' book, 2011

- Quasi beam function :



Lorentz boost and $L \to \infty$

$$y_P - y_B = y_{\tilde{P}}$$



$$n_b^\mu(y_B) = (n_b^+, n_b^-, \vec{0}_\perp) = (-e^{2y_B},1,\vec{0}_\perp)$$

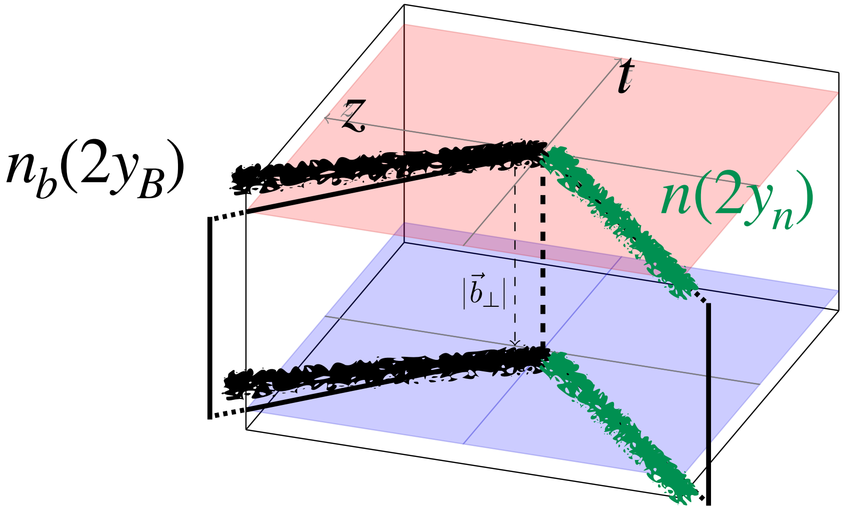
Spacelike but close-to-lightcone ($y_B \to -\infty$) Wilson lines, not calculable on the lattice ☹

Equal-time Wilson lines, directly calculable on the lattice🙂

Ebert, Schindler, Stewart and YZ, JHEP 04 (2022).

Soft function (in Collins' scheme)

- Not directly calculable, but has the asymptotic behavior:

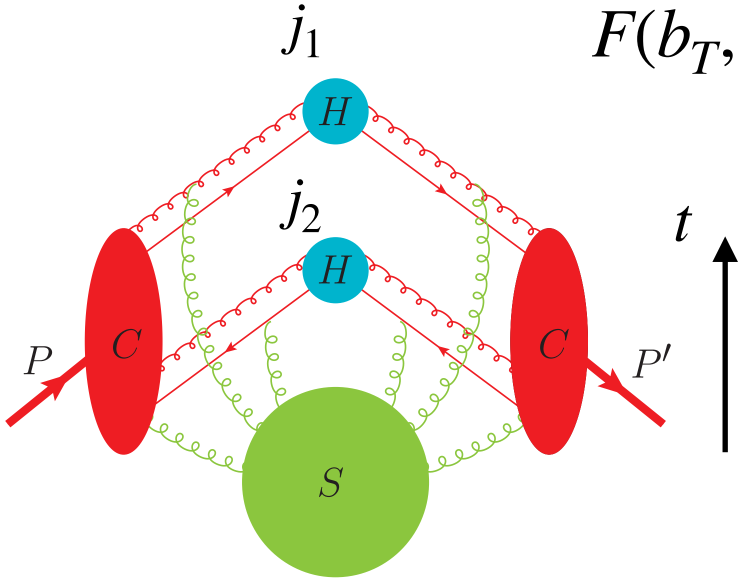


Collins-Soper kernel

$$\xrightarrow{y_n - y_B \to \infty} S_r(b_T, \mu)\; e^{-2(y_n - y_B)\gamma_\zeta(b_T,\mu)}$$

Reduced soft factor

- $S_r(b_T, \mu)$ can be extracted from a meson form factor:



$$F(b_T, P^z) = \langle \pi(-P) \,|\, j_1(b_T) j_2(0) \,|\, \pi(P) \rangle$$

$$\overset{P^z \gg m_\pi}{=} S_r(b_T, \mu) \int dx dx' \; H(x, x', \mu) \Phi^\dagger(x, b_T, P^z, \mu) \Phi(x', b_T, P^z, \mu)$$

Φ: quasi-TMD wave function

- Ji, Liu and Liu, NPB 955 (2020), PLB 811 (2020);

Factorization formula for the quasi-TMDs

(Subtracted) quasi-TMDPDF:

$$\frac{\tilde f_i(x,\mathbf{b}_T,\mu,y_{\tilde P}=y_n-y_B)}{\sqrt{S^q(b_T,\mu,2(y_n-y_B))}} = \lim_{-y_B\gg1} \frac{\lim_{\epsilon\to0} Z^B_{\rm UV}(\epsilon,\mu)B_i(x,b_T,\epsilon,y_P-y_B)}{\sqrt{\lim_{\epsilon\to0} Z^S_{\rm UV}(b_T,\epsilon,\mu,2(y_n-y_B))S^q(b_T,\epsilon,2(y_n-y_B))}}$$

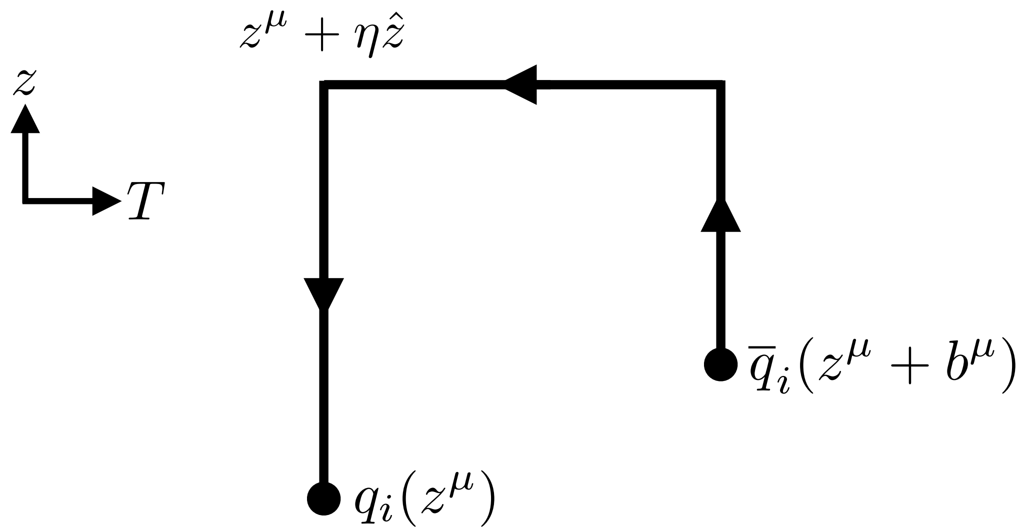
Differing from the TMDPDF by the order of UV renormalization and $P^z \to \infty$ limit, which can be matched in LaMET:

$$\tilde f_{i/p}(x,\mathbf{b}_T,\mu,\tilde P^z)\sqrt{S_I(b_T,\mu)} = C\left(\mu,x\tilde P^z\right)\ \exp\left[\frac{1}{2}\gamma_\zeta(\mu,b_T)\ln\frac{(2x\tilde P^z)^2}{\zeta}\right]$$
$$\times f_{i/p}(x,\mathbf{b}_T,\mu,\zeta)\left\{1+\mathcal{O}\left[\frac{1}{(x\tilde P^z b_T)^2},\frac{\Lambda^2_{\rm QCD}}{(x\tilde P^z)^2}\right]\right\}$$

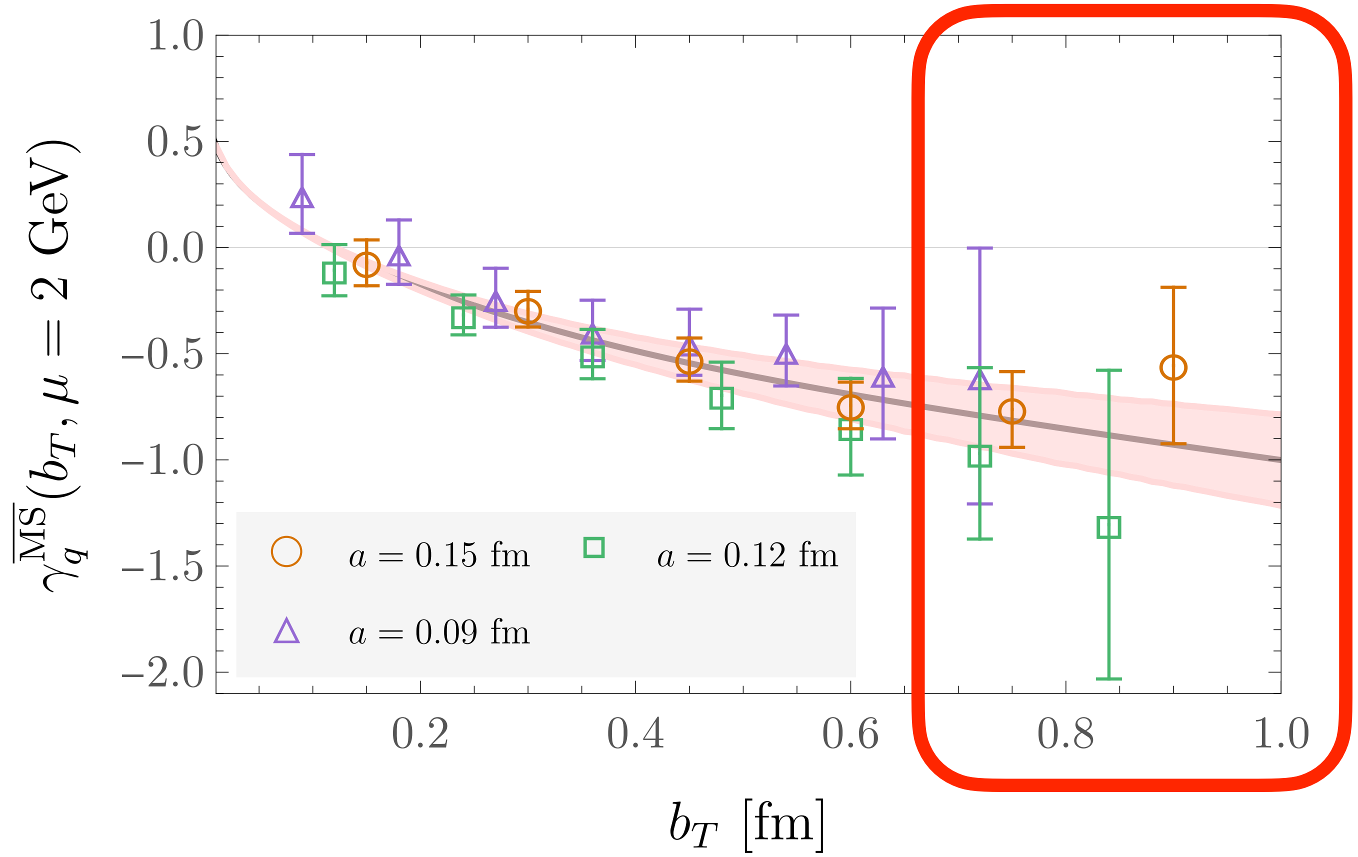
- Ji, Liu and Liu, NPB 955 (2020), PLB 811 (2020);
- Ebert, Stewart, YZ, PRD99 (2019), JHEP09 (2019);
- Ebert, Schindler, Stewart and YZ, JHEP 04 (2022).

Some key systematics in lattice calculation

Staple-shaped Wilson line



$$\eta \gg \{b^z, b_T\},\ xP^z \gg 1/b_T$$

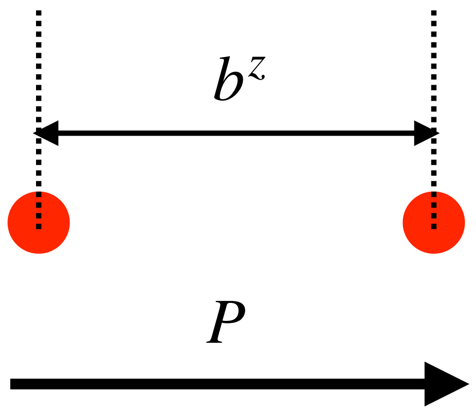


- Gauge link induces statistical noise, while signal is exponentially suppressed at large b_T due to Wilson line self-energy;
- Complex operator mixings due to the breaking of symmetries by the staple;
- Additional systematics due to multiple scales $\{b^z, b_T, \eta\}$ involved.

Quasi-TMD in the Coulomb gauge

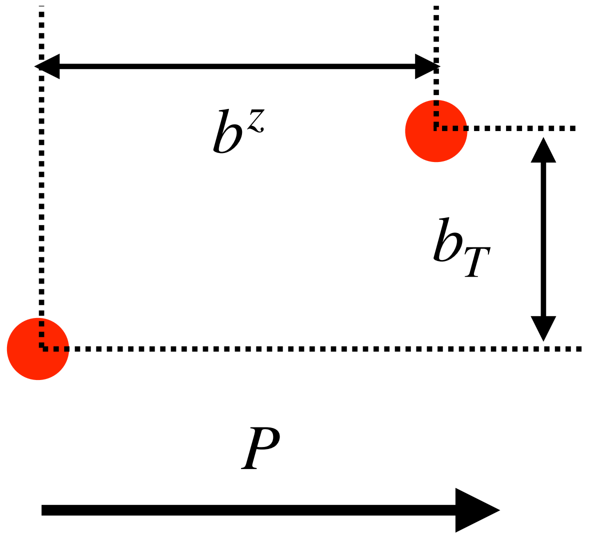
$$\tilde{f}(x, b_T, P^z, \mu) = \frac{P^z}{P^t}\int_{-\infty}^{\infty}\frac{db^z}{4\pi}\, e^{ixP^zb^z}\langle P|\bar{\psi}(\vec{b})\gamma^t\psi(0)\Big|_{\nabla\cdot\mathbf{A}=0}|P\rangle$$

Quasi-PDF



X. Gao, W.-Y. Liu and YZ, PRD 109 (2024).

Quasi-TMD



YZ, PRL 133 (2024).

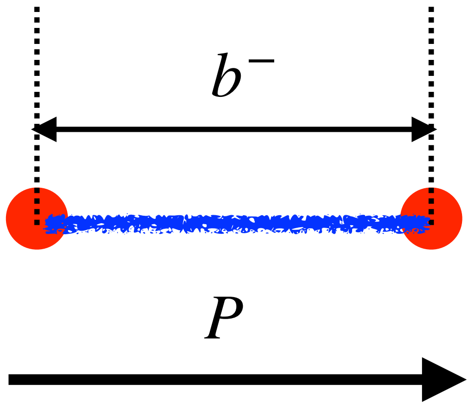
“Universality Class”

- Y. Hatta, X. Ji, and YZ, PRD 89 (2014);
- X. Ji, Y.-S. Liu, Y. Liu, J.-H. Zhang and YZ, RMP 93 (2021).

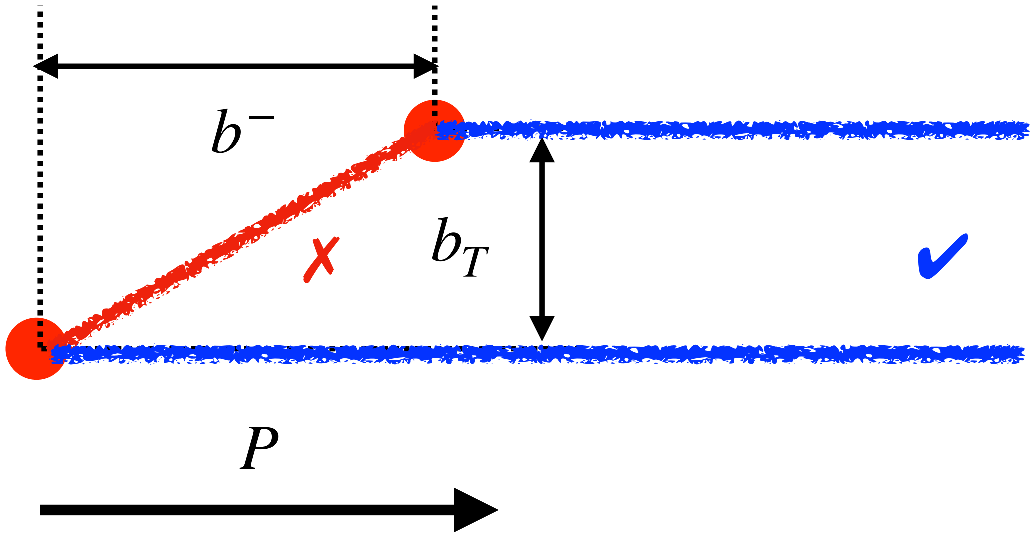
$$G(A) = 0, \quad G(A) = A^0, A^z, \nabla\cdot\mathbf{A}, A^+$$

Quasi-TMD under the infinite boost

$$\tilde{f}(x, b_T, P^z, \mu) = \frac{P^z}{P^t}\int_{-\infty}^{\infty}\frac{db^z}{4\pi}\, e^{ixP^z b^z}\langle P\,|\,\bar{\psi}(\vec{b})\gamma^t\psi(0)\Big|_{\nabla\cdot\mathbf{A}=0}\,|\,P\rangle$$



PDF



TMD

“Universality Class”

- Y. Hatta, X. Ji, and YZ, PRD **89** (2014);
- X. Ji, Y.-S. Liu, Y. Liu, J.-H. Zhang and YZ, RMP **93** (2021).

$$G(A) = 0, \quad G(A) = A^0, A^z, \nabla\cdot\mathbf{A}, A^+$$

Quasi-TMD under the infinite boost

Gauge-invariant extension of the Coulomb gauge:

- P. A. M. Dirac, Can. J. Phys. **33** (1955);
- M. Lavelle and D. McMullan, Phys. Rept. **297** (1997).

$$\Psi_C(x) \equiv U_C(x)\psi(x), \qquad \vec{\nabla}\cdot\left[U_C\vec{A}U_C^{-1} + \frac{i}{g}U_C\vec{\nabla}U_C^{-1}\right] = 0$$

Under arbitrary compact gauge transformation $U(x)$,

$$\psi(x) \to U\psi(x), \quad U_C \to U_C U^{-1}, \quad \Psi_C(x) \to \Psi_C(x), \quad \bar{\psi}_C(\vec{b})\gamma^t\psi_C \to \bar{\psi}_C(\vec{b})\gamma^t\psi_C$$

Compact perturbative solution:

$$U_C = \sum_{n=0}^{\infty}\frac{(ig)^n}{n!}\omega_n \qquad \omega_1 = -\frac{1}{\nabla^2}\vec{\nabla}\cdot\vec{A},$$

$$\frac{\omega_2}{2!} = \frac{1}{\nabla^2}\left(\vec{\nabla}\cdot\left(\omega_1^\dagger\vec{\nabla}\omega_1\right) - [\vec{\nabla}\omega_1, \cdot\vec{A}]\right) ,$$

$$\ldots$$

$$U_C(A)\Big|_{\vec{\nabla}\cdot\vec{A}=0} = 1$$

Quasi-TMD under the infinite boost

- Infinite boost limit along the *z* direction:

$$-\frac{1}{\nabla^2}\vec{\nabla}\cdot\vec{A}(x) = i\int\frac{d^4k}{(2\pi)^4}\,e^{-ik\cdot x}\frac{1}{k_z^2+k_\perp^2}\left[k^z\tilde{A}^z(k)+k_\perp\cdot\tilde{A}_\perp(k)\right]$$

$$\approx i\int\frac{d^4k}{(2\pi)^4}\,e^{-ik\cdot x}\frac{k^+}{(k^+)^2+\epsilon^2}\tilde{A}^+(k)$$

$$=\frac{1}{2}\left[\int_{-\infty^-}^{x^-}+\int_{+\infty^-}^{x^-}\right]d\eta^-\;A^+(x^+,\eta^-,x_\perp)$$

$$\equiv\frac{1}{\partial^+_{\rm pv}}A^+(x)$$

Principle-value (P.V.) prescription:

$$\frac{k^+}{(k^+)^2+\epsilon^2}=\frac{1}{2}\left[\frac{1}{k^++i\epsilon}+\frac{1}{k^+-i\epsilon}\right]$$

Past pointing

Future pointing

$$\frac{\omega_n}{n!}\to\frac{1}{\partial^+_{\rm pv}}\left(\ldots\left(\frac{1}{\partial^+_{\rm pv}}\left((\frac{1}{\partial^+_{\rm pv}}A^+)A^+\right)A^+\right)\ldots A^+\right)$$

Path-ordered integral

$$U_C\to\mathscr{P}\exp\left[-ig\int_{x^-}^{\mp\infty^-}dy^-A^+(y^-)\right]\equiv W_n^\dagger(x,\mp\infty^-)$$

Infinite "P.V. prescribed" light-like Wilson line

Factorization formula

Factorization of the quasi beam function (derived with SCET):

$$\tilde{B}(x, b_\perp, \mu, \tilde{P}^z) = |\, C(x\tilde{P}^+/\mu)\,|^2\, B(x, b_\perp, \ldots, x\tilde{P}^+) S_C^0(b_\perp, \ldots) + O(\Lambda_{\rm QCD}/\tilde{P}^z)$$

"…", UV and rapidity scales

Quasi beam function: $\tilde{B}(x, b_\perp, \mu, \tilde{P}^z)$

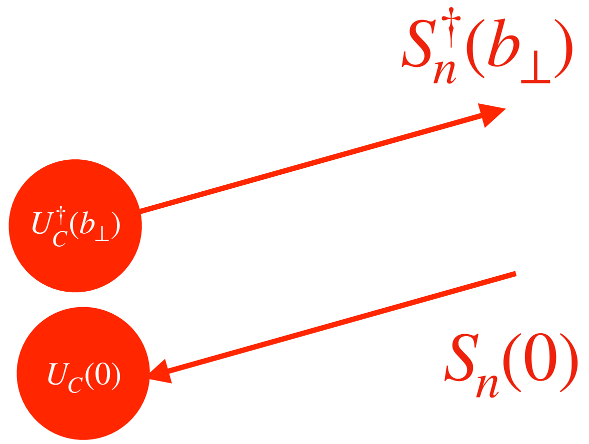
- Soft and collinear modes
- Gribov ambiguities

Zero-bin-subtracted beam function: $B(x, b_\perp, \ldots, x\tilde{P}^+)\,, \qquad 0 < x < 1$

- (Hard) collinear modes only
- No Gribov ambiguity

Quasi-zero-bin: $S_C^0 = \dfrac{1}{N_c} \langle 0 \,|\, T\left[S_n^\dagger(b_\perp)(U_C^s)^\dagger(b_\perp)U_C^s(0)S_n(0)\right] |\, 0 \rangle$

- Soft modes only
- Gribov ambiguities



Not directly calculable on the lattice

Factorization formula

$$\tilde{B}(x, b_\perp, \mu, \tilde{P}^z) = |C(x\tilde{P}^+/\mu)|^2 B(x, b_\perp, \ldots, x\tilde{P}^+) S_C^0(b_\perp, \ldots) + O(\Lambda_{\rm QCD}/P^z)$$

Physical (or subtracted) TMDPDF:

$$f(x, b_\perp, \mu, \tilde{\zeta}) = B(x, b_\perp, \ldots, x\tilde{P}^+) S(b_\perp, \ldots, y_n)$$

$$\downarrow$$

Standard soft factor

Collins-Soper scale

$$\tilde{\zeta} = 2(x\tilde{P}^+)^2 e^{-2y_n}$$

Gribov ambiguities cancel!

$$\boxed{\frac{\tilde{B}(x, b_\perp, \mu, \tilde{P}^z)}{\tilde{S}_C(b_\perp, \mu, y_n)}} = |C(x\tilde{P}^+/\mu)|^2 f(x, b_\perp, \mu, \tilde{\zeta}) + O(\lambda)$$

Quasi soft factor:

$$\tilde{S}_C(b_\perp, \mu, y_n) \equiv \frac{S_C^0(b_\perp, \ldots)}{S(b_\perp, \ldots, y_n)}$$

$$\Rightarrow$$

$$\frac{\tilde{B}(x, b_\perp, \mu, \tilde{P}^z)}{\tilde{S}_C(b_\perp, \mu, 0)} = |C(x\tilde{P}^+/\mu)|^2 \exp\left[\frac{1}{2}\gamma_\zeta(b_\perp, \mu) \ln \frac{2(x\tilde{P}^+)^2}{\zeta}\right] \times f(x, b_\perp, \mu, \zeta) + O(\lambda)$$

Same form as the gauge-invariant quasi-TMD factorization

$$\frac{d}{dy_n} \ln \tilde{S}_C(b_\perp, \mu, y_n) = \gamma_\zeta(b_\perp, \mu)$$

One-loop quasi soft function

- Collins regulator: $n_A^\mu = (n_A^+, n_A^-, n_A^\perp) = (1, -e^{-2y_A}, 0_\perp)\,, \qquad y_A \gg 1\,,$

$$n_B^\mu = (n_B^+, n_B^-, n_A^\perp) = (-e^{2y_B}, 1, 0_\perp)\,, \qquad y_B \ll -1$$

$$\tilde{S}_C(b_\perp, \mu, y_n) \equiv \lim_{y_A \to \infty} \frac{S_C^0(b_\perp, \mu, y_A)}{S(b_\perp, \mu, y_A - y_n)} = 1 + \frac{\alpha_s C_F}{2\pi}(1 - 2y_n)\mathbf{L}_b$$

$$\mathbf{L}_b = \ln \frac{b_T^2 \mu^2 e^{2\gamma_E}}{4}$$

- η regulator:

$$W_n = \sum_{\text{perm}} \exp\left[-\frac{g}{\bar{n} \cdot \partial} \frac{|\bar{n} \cdot \partial|^{-\eta}}{\nu^\eta} \bar{n} \cdot A_n(0)\right]$$

$$\tilde{S}_C(b_\perp, \mu, y_n) \equiv \lim_{y_A \to \infty} \frac{S_C^0(b_\perp, \mu, y_A)}{S(b_\perp, \mu, y_A - y_n)} = 1 + \frac{\alpha_s C_F}{2\pi}\mathbf{L}_b \qquad y_n = 0$$

Rapidity regulator independent!

Quasi soft factor

Can be extracted from the same meson form factor:

$$
\begin{aligned}
F(b_\perp, P^z) &= \langle \pi(-P) | j_1(b_\perp) j_2(0) | \pi(P) \rangle \\
&= \int dx_1 dx_2 \; H_F(x_1, x_2, P^z, \mu) \\
&\quad \times \phi^\dagger(x_1, b_\perp, \mu, P^+, y_n) \phi(x_2, b_\perp, \mu, P^+, -y_n)
\end{aligned}
$$

Hard kernel: known at 1-loop

$$\frac{\tilde{\phi}(x, b_\perp, \mu, P^z)}{\tilde{S}(b_\perp, \mu, y_n)} = C\left(\frac{xP^+}{\mu}\right) C\left(\frac{\bar{x}P^+}{\mu}\right) \phi_{\rm pv}(x, b_\perp, \mu, P^+, y_n)$$

$$
\begin{aligned}
F(b_\perp, P^z) = \int dx_1 dx_2 \; &\frac{H_F(x_1, x_2, P^z, \mu)}{C(x_1)C(x_2)C(\bar{x}_1)C(\bar{x}_2)} \\
&\times \frac{\tilde{\phi}(x_1, b_\perp, \mu, P^z)}{\tilde{S}_C(b_\perp, \mu, 0)} \frac{\tilde{\phi}(x_2, b_\perp, \mu, P^z)}{\tilde{S}_C(b_\perp, \mu, 0)}
\end{aligned}
$$

$S_I^C(b_T, \mu) = 1/[\tilde{S}_C(b_T, \mu, 0)]^2$

$\tilde{\phi}(x, b_T, \mu, P^z)$: Coulomb gauge quasi-TMD wave function ✓

$\phi^* = \phi$ due to the P.V. prescription

Lattice calculation

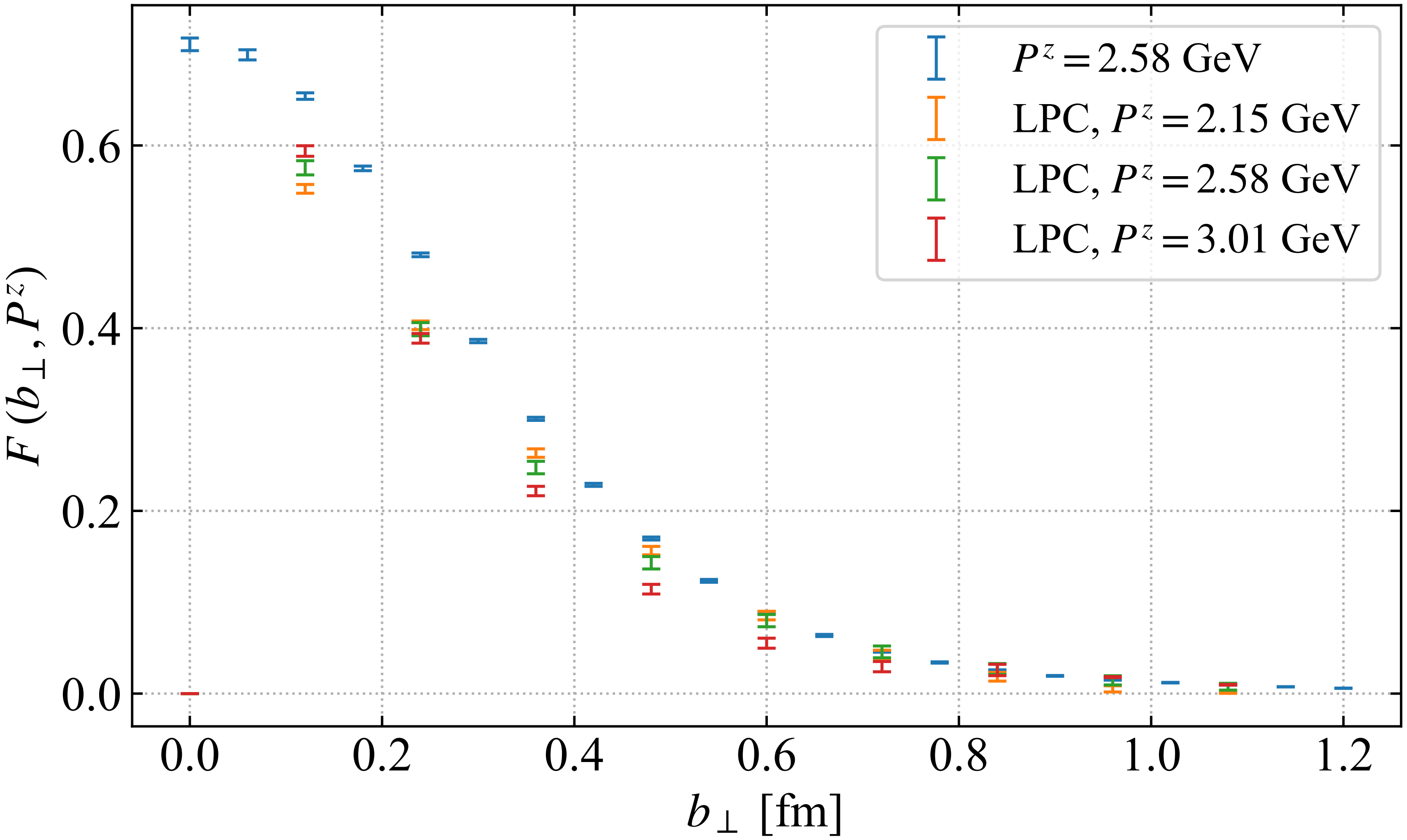
- N_f=2+1 HotQCD configurations with the Highly Improved Staggered Quark (HISQ) action;
- Lattice spacing a=0.06 fm;
- Lattice volume, 48^3x64;
- Valence pion mass, 670 MeV.

Observable	# of configs	$P^z_{\max}$
Form factor	100	2.58 GeV
Quasi-TMD WF	553	4.30 GeV

Form factor

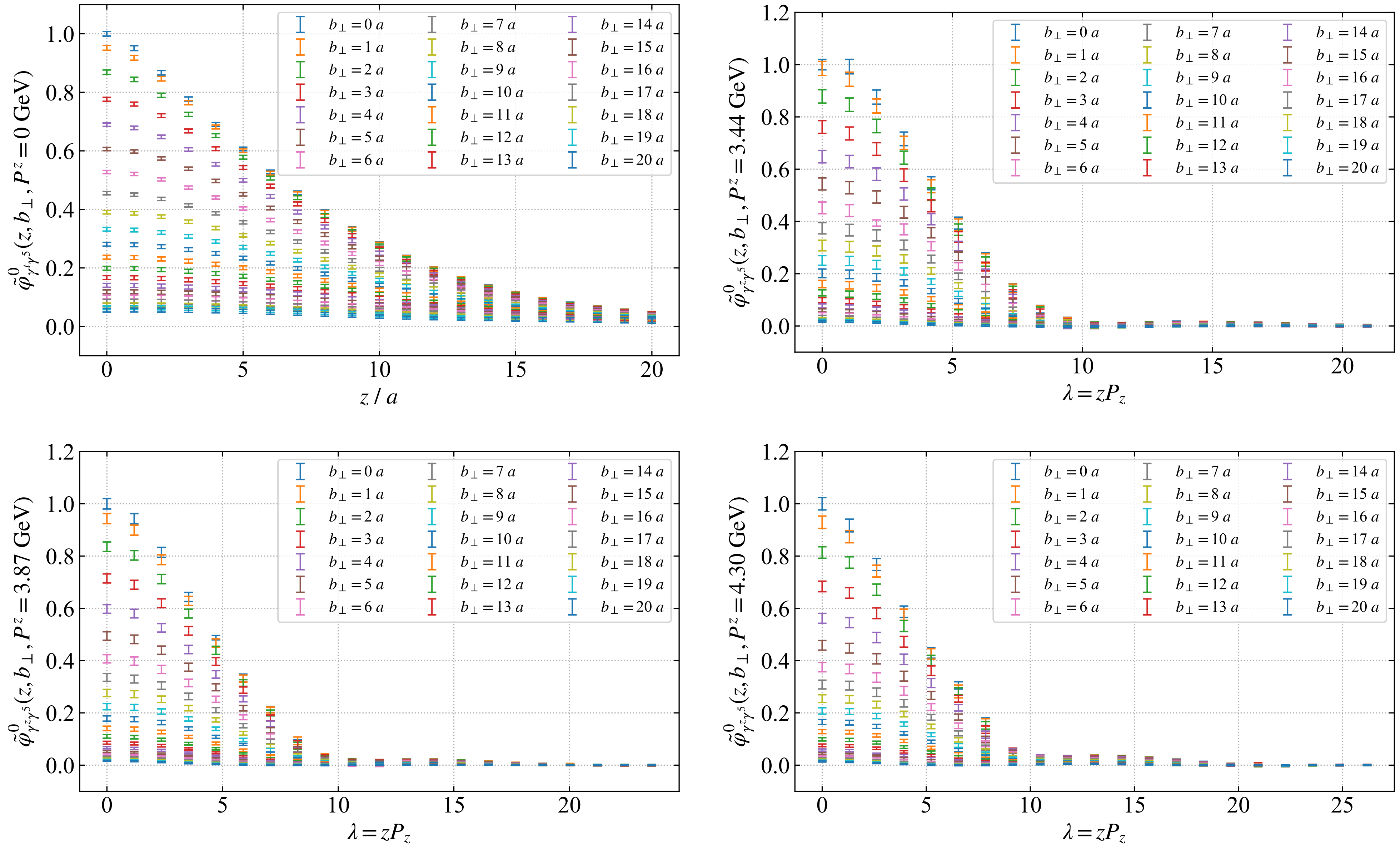
- Generated with kinematically enhanced pion interpolators

R. Zhang, A. Grebe, D. Hackett, M. Wagman and YZ, PRD 112 (2025) 5, L051502.

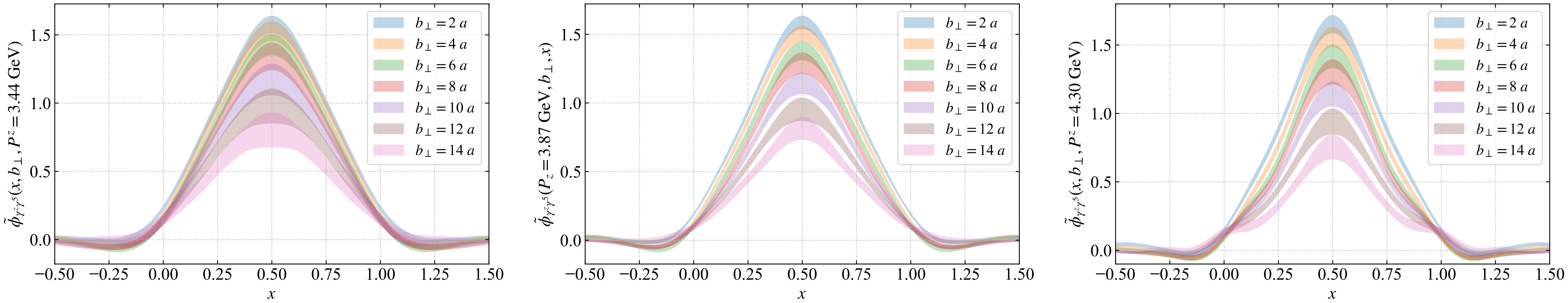


LPC, JHEP 08 (2023) 172, MILC ensembles, a=0.12, 0.10 fm, pion mass 310 MeV.

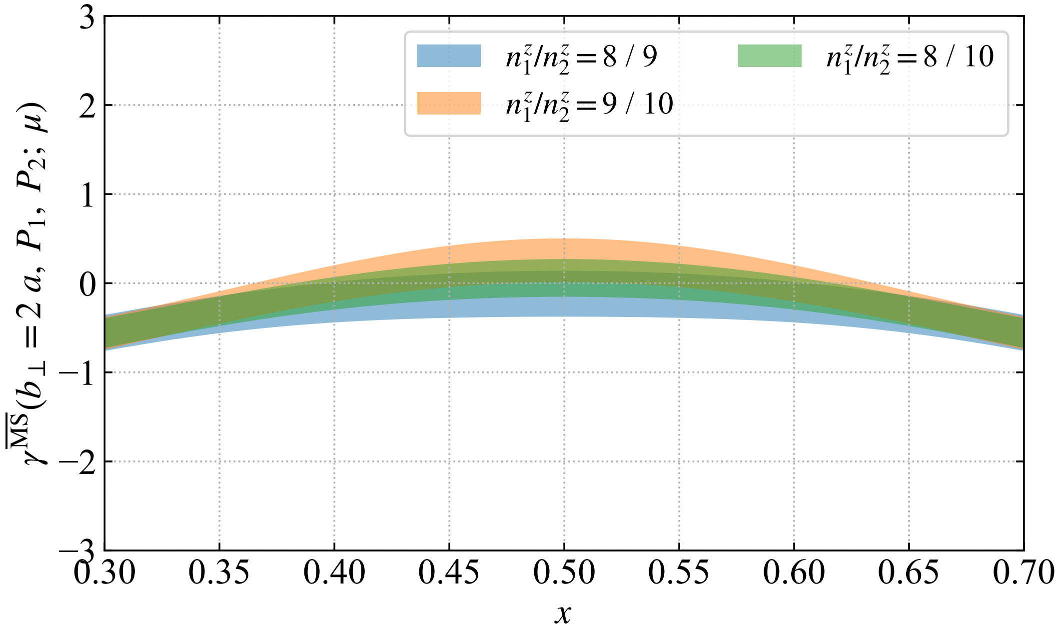
Quasi-TMD wave function

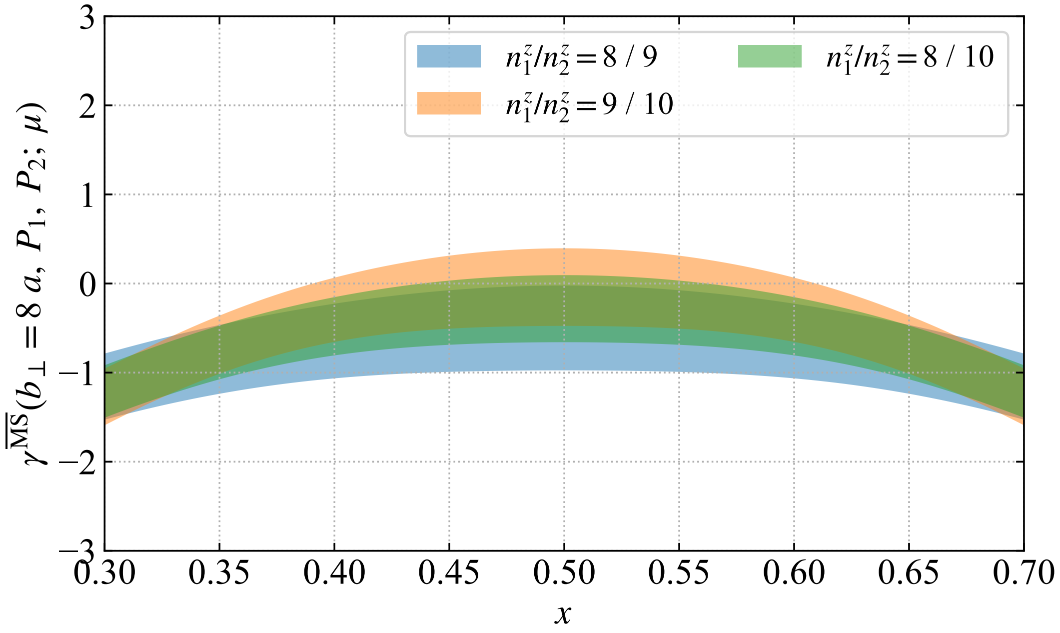


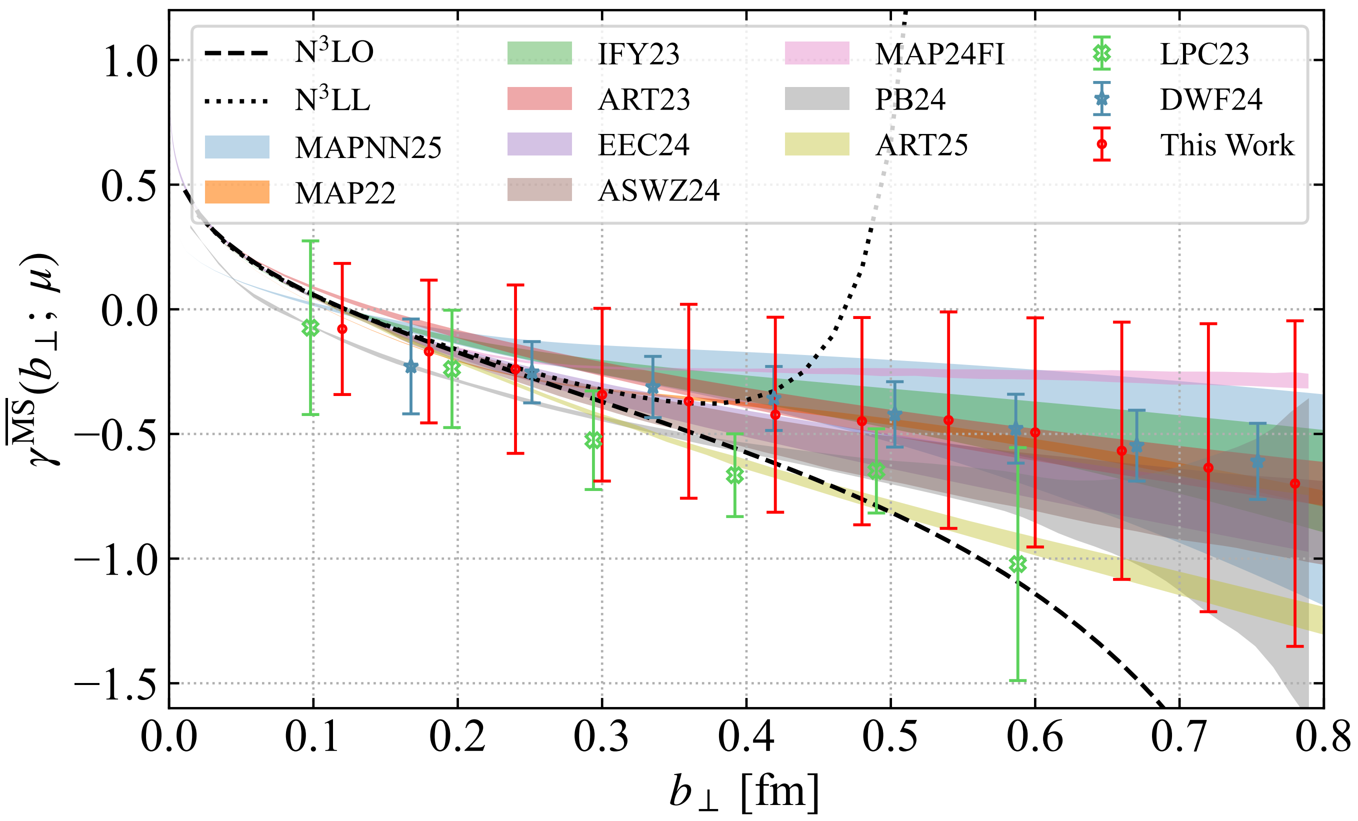
Quasi-TMD Wave function



Collins-Soper kernel







Intrinsic soft factor

- Renormalized quasi-TMD WF:

$$\tilde{\varphi}^{\text{ratio}}\left(z, b_\perp, P^z; \mu\right) = \frac{\tilde{\varphi}^{0}\left(z, b_\perp, P^z; a\right)}{\tilde{\varphi}^{0}\left(z=0, b_\perp, P^z=0; a\right)} = \frac{\tilde{\varphi}^{\overline{\text{MS}}}\left(z, b_\perp, P^z; \mu\right)}{\tilde{\varphi}^{\overline{\text{MS}}}\left(z=0, b_\perp, P^z=0; \mu\right)}$$

$$S_I^{\text{ratio}}(b_\perp; \mu) = S_I^{\overline{\text{MS}}}(b_\perp; \mu) \cdot \left| \tilde{\varphi}^{\overline{\text{MS}}}\left(z=0, b_\perp, P^z=0; \mu\right) \right|^2$$

- At small b_T,

$$S_I^{\text{ratio}}(b_\perp; \mu) = S_I^{\overline{\text{MS}}}(b_\perp; \mu) \left| C_0(b_T, \mu) \right|^2$$

$C_0(b_T, \mu)$: Wilson coefficient of zero momentum matrix element

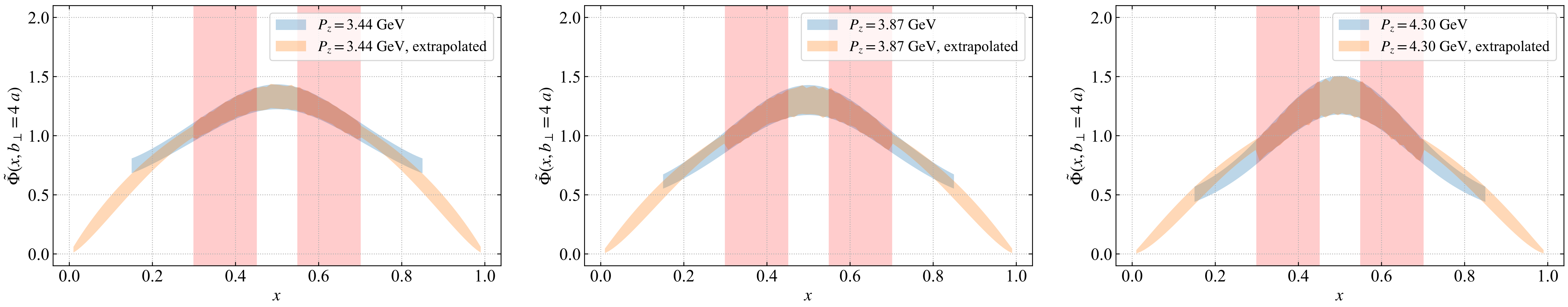
Intrinsic soft factor

- Convolutional integral:

$$F(b_\perp, P^z) = S_I^{\text{ratio}}(b_T, \mu) \int_0^1 dx_1 \int_0^1 dx_2 \; \frac{H_F(x_1, x_2, P^z, \mu)}{C(x_1)C(x_2)C(\bar{x}_1)C(\bar{x}_2)}$$

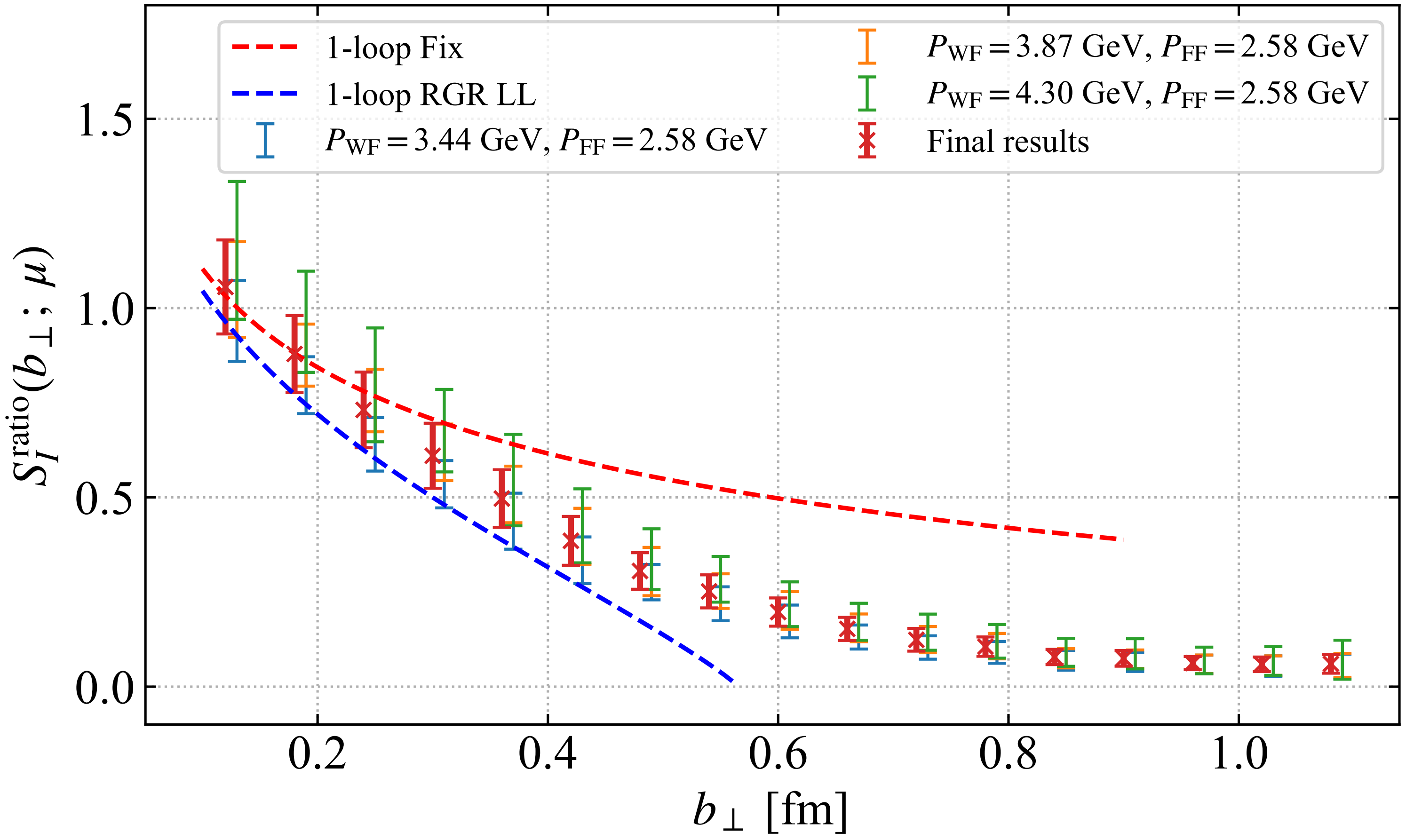
$$\times\, \tilde{\phi}^{\text{ratio}}(x_1, b_\perp, \mu, P^z)\tilde{\phi}^{\text{ratio}}(x_2, b_\perp, \mu, P^z)$$

- Extrapolating the end-point regions of the quasi-TMDWF



Intrinsic soft factor

- Evolving all quasi-TMD WFs to P^z=2.58 GeV using the calculated CS kernel;
- Numerical integration within 0.05 < x < 0.95 with small cutoff effects.



Summary

- The Coulomb-gauge quasi-TMDs provide a new way to calculate TMDs from lattice QCD;
- The operator definition of the quasi-soft factor is derived using SCET;
- The quasi-soft factor (or intrinsic soft function) can be obtained from the same form factor in the traditional approach;
- A fine lattice, heavy quark mass calculation of the intrinsic soft function shows nice agreement with perturbation theory at small bT.
- Future improvement: physical quark masses, continuum limit, higher-order perturbative corrections, etc.