Lattice Parton Collaboration

SHANGHAI JIAO TONG UNIVERSITY

Collins-Soper Kernel & Intrinsic Soft Function from Lattice QCD

Zhi-Chao Gong
Shanghai Jiao Tong University
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In Collaboration with

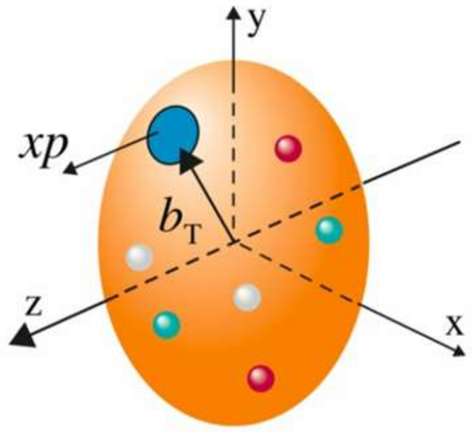
Min-Huan Chu, Jin-Xin Tan, Han-Zhang Wang, Wei Wang, Qi-An Zhang

Content

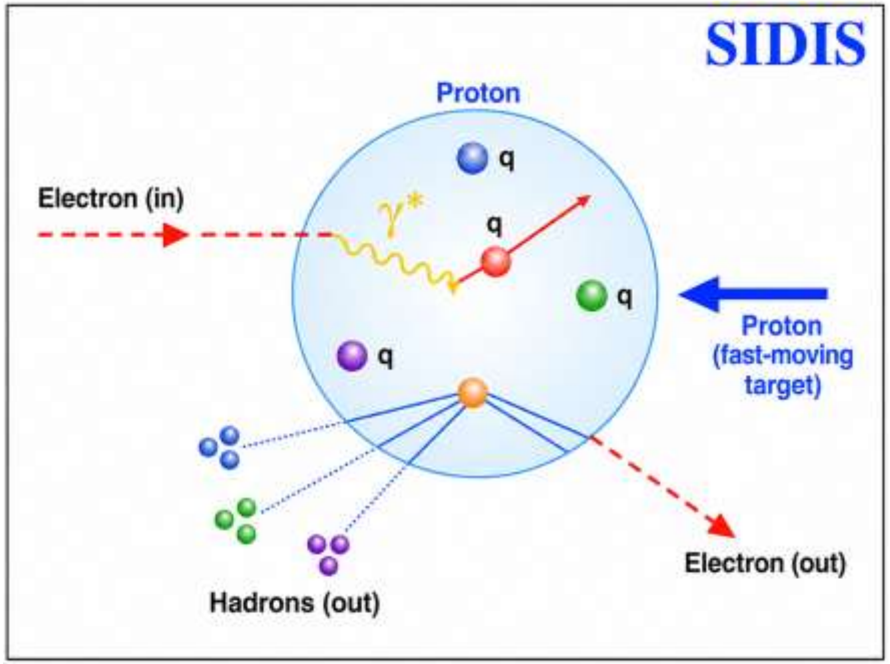
- Motivation and Theoretical Framework
- Lattice Calculation
- Summary and Outlook

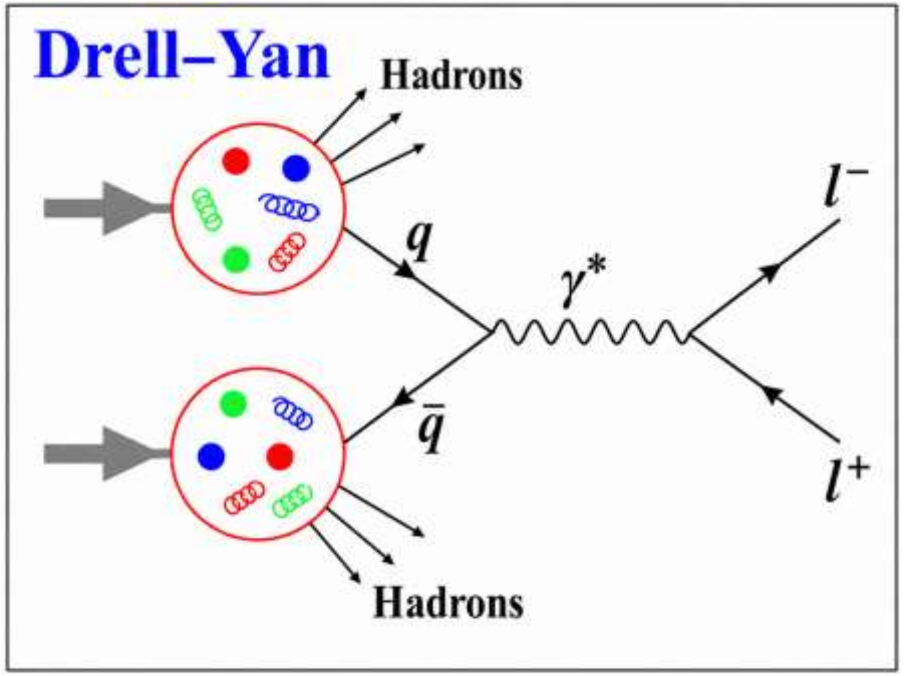
Content

TMDs and 3D Hadron Structure



TMDs





TMDs are key inputs for SIDIS and Drell-Yan process!

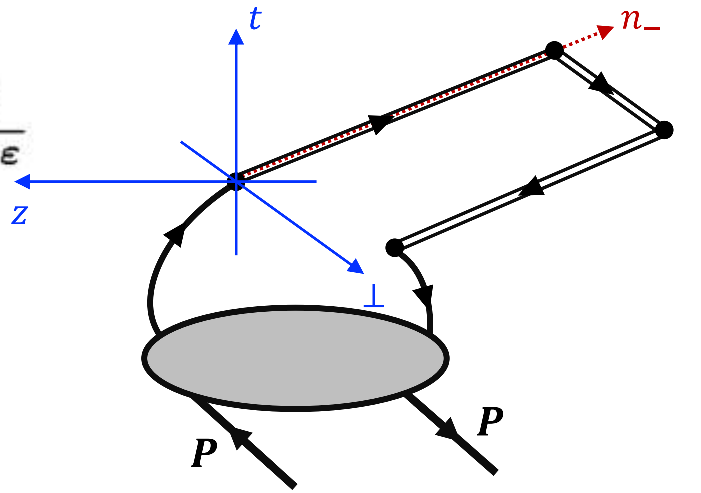
Boussarie et al., TMD Handbook, 2304.03302

Definitions

➢ **Rapidity divergence :**

$$I_{\text{div}} \sim \int dk^+ dk^- \frac{f(k^+k^-)}{(k^+k^-)^{1+\varepsilon}} = \frac{1}{2}\int \frac{d(k^-/k^+)}{k^-/k^+} \int d(k^+k^-) \frac{f(k^+k^-)}{(k^+k^-)^{1+\varepsilon}}$$

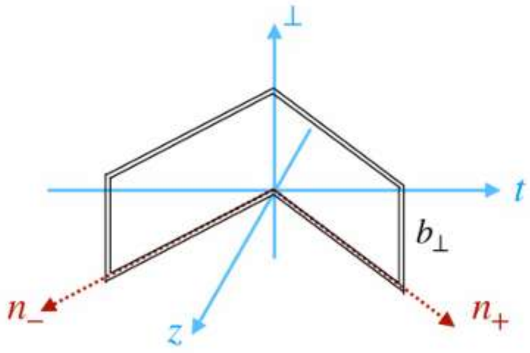
$y_k = \frac{1}{2}\ln\frac{k^+}{k^-}$ **$y_k \to \pm\infty$: rapidity divergence**



➢ **Rapidity divergence subtracted by soft function:**

$$S(b_\perp, \mu, Y) = \frac{1}{N_c} \mathrm{Tr}\langle 0| W_n(\vec{b}_\perp; Y)\, W_p^\dagger(\vec{b}_\perp; Y)|0\rangle$$

Y as the rapidity cutoff.



Ji et al., PRD 71, 034005(2005);
Collins, Vol. 32(Cambridge University Press, 2011).

Definitions

Ji et al., PRD 71, 034005(2005);
Collins, Vol. 32(Cambridge University Press, 2011).

- The **intrinsic/reduced soft function** is defined from

$$S(b_\perp, \mu, Y) = S_I(b_\perp, \mu) \exp\left[2Y K(b_\perp, \mu)\right]$$

- **Rapidity scale evolution** controlled by :

$$\frac{\partial}{\partial Y} \ln S(b_\perp, \mu, Y) = 2K(b_\perp, \mu)$$

→ **Collins-Soper kernel**

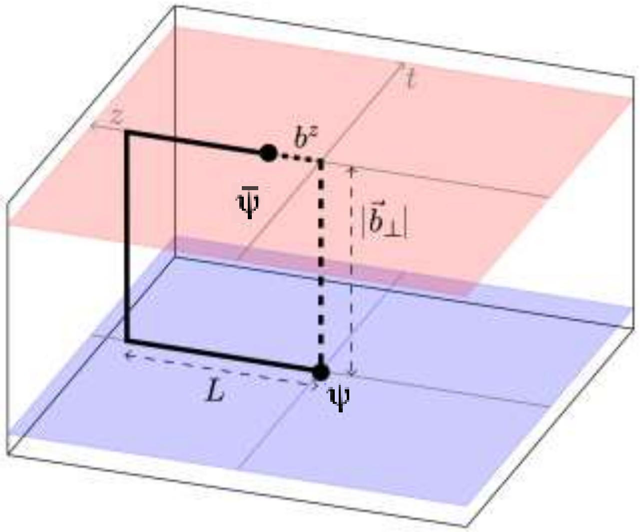
- Collins-Soper equation:

$$\frac{\partial}{\partial \ln \sqrt{\zeta}} \ln f_{\text{TMD}}(x, b_\perp, \mu, \zeta) = K(b_\perp, \mu)$$

CS kernel and S_I are essential in describing the rapidity dependence of TMDs.

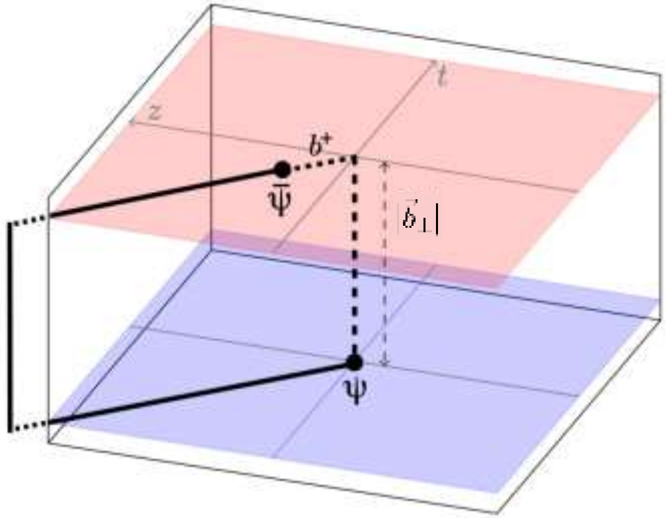
Lattice & LaMET

LaMET gives a bridge between light-cone quantities and lattice-calculable quantities.



Lorentz boost

$L \to \infty$
$P^z \to \infty$



For TMDs

Perturbative matching kernel

Light-cone TMDs

$$\tilde{f}_\Gamma(x, b_\perp, \mu, \zeta^z) S_I^{\frac{1}{2}}(b_\perp, \mu) = H_\Gamma(\zeta^z, \mu) \exp\left[\frac{1}{2} K(b_\perp, \mu) \ln \frac{\zeta^z - i\epsilon}{\zeta}\right] f(x, b_\perp, \mu, \zeta) + \mathcal{O}\left(\frac{\Lambda_{\rm QCD}^2}{\zeta^z}, \frac{M^2}{(P^z)^2}, \frac{1}{b_\perp^2 \zeta^z}\right)$$

Quasi-TMDs

Ji, PLB811(2020); Ebert, JHEP04(2022)

Lattice & LaMET

For CS kernel:

Quasi-TMDs

$$K(b_\perp,\mu)=\frac{1}{\ln(P_1^z/P_2^z)}\ln\left(\frac{H_\Gamma(\zeta_2^z,\mu)\tilde{f}_\Gamma(x,b_\perp,\zeta_1^z,\mu)}{H_\Gamma(\zeta_1^z,\mu)\tilde{f}_\Gamma(x,b_\perp,\zeta_2^z,\mu)}\right)$$

For a better x-plateau, quasi-TMDWFs are used here.

Chu et al.(LPC),JHEP 08 (2023) 172

For intrinsic soft function

Form factor

Ji et al. , Nucl.Phys.B 955 (2020)

$$S_I(b_\perp,\mu)=\frac{F(b_\perp,P_1,P_2,\mu)}{\int \mathrm{d}x_1\mathrm{d}x_2 H(x_1,x_2)\tilde{\Phi}^\dagger(x_2,\zeta_2^z,b_\perp,\mu)\tilde{\Phi}(x_1,\zeta_1^z,b_\perp,\mu)}$$

matching kernel

Quasi-TMDWF

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Lattice Setup

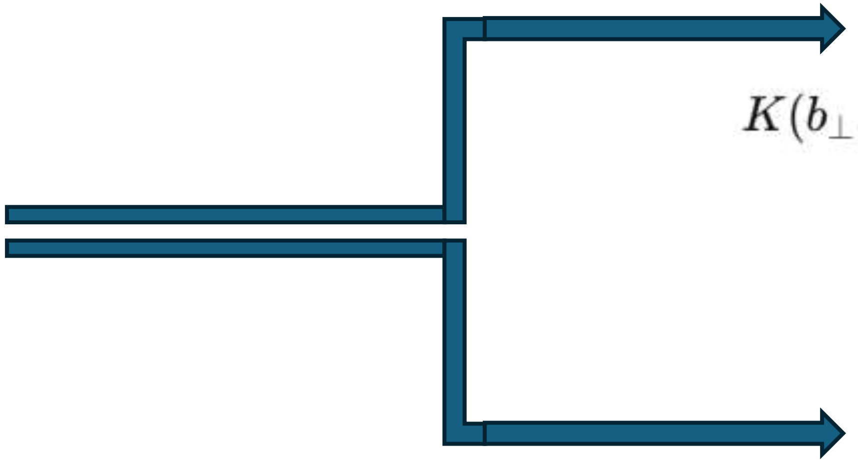
	a/fm	m_π/MeV	P^z/GeV	$n_{cfg} \times n_{meas}$	
				CS Kernel	S_I
C32P29		292.4	1.47, 1.84, 2.21	984×4	984×2
C32P23	0.10530	228.0	1.47, 1.84, 2.21	448×10	450×24
C48P14		**135.5**	0.98, 1.22, 1.47	204×24	$204 \times 8 \times 12$
F32P30	0.07746	303.2	2.00, 2.50, 3.00	1153×6	1153×6
H48P32	**0.05187**	317.2	2.00, 2.50, 3.00	550×8	550×8

Ensembles with multi lattice spacings and pion masses are used.

Du et al. (CLQCD) , Phys.Rev.D 111 (2025).

Main Strategy

Quasi-TMDWF



Collins-Soper Kernel

$$K(b_\perp,\mu)=\frac{1}{\ln\left(P_1^z/P_2^z\right)}\ln\left[\frac{H_\Gamma\left(\zeta_2^z,\mu\right)\tilde{\Phi}\left(x,b_\perp,\zeta_1^z,\mu\right)}{H_\Gamma\left(\zeta_1^z,\mu\right)\tilde{\Phi}\left(x,b_\perp,\zeta_2^z,\mu\right)}\right]$$

Intrinsic Soft Function

$$S_I(b_\perp,\mu)=\frac{F(b_\perp,P_1,P_2,\mu)}{\int \mathrm{d}x_1\mathrm{d}x_2 H(x_1,x_2)\tilde{\Phi}^\dagger(x_2,\zeta_2^z,b_\perp,\mu)\tilde{\Phi}(x_1,\zeta_1^z,b_\perp,\mu)}$$

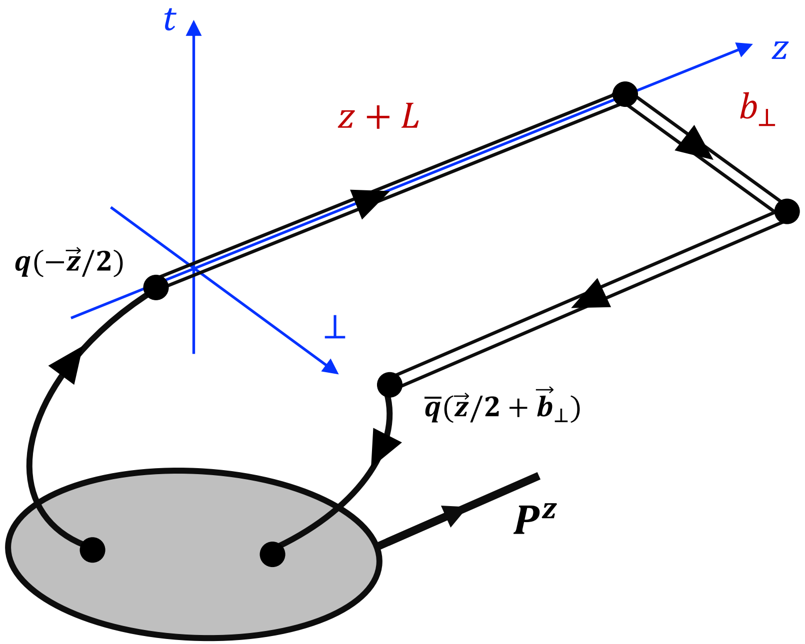
Quasi-TMDWF is a common lattice input of two quantities.

Quasi-TMDWF

Quasi-TMD Wave Function

➢ **Bare quasi-TMDWF in coordinate space**

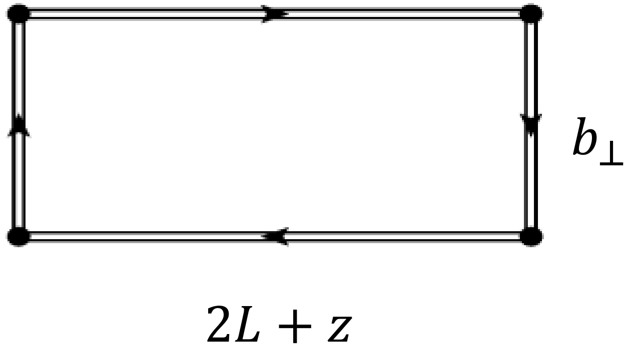
$$\tilde{\Phi}^{\pm 0}(z, b_\perp, P^z, L) = \langle 0|\bar{q}(z\hat{n}_z/2 + b_\perp\hat{n}_\perp)\Gamma U_{\sqsupset,\pm}(L, z, b_\perp) q(-z\hat{n}_z/2)|P^z\rangle.$$



➢ **Subtracted quasi-TMDWF in momentum space**

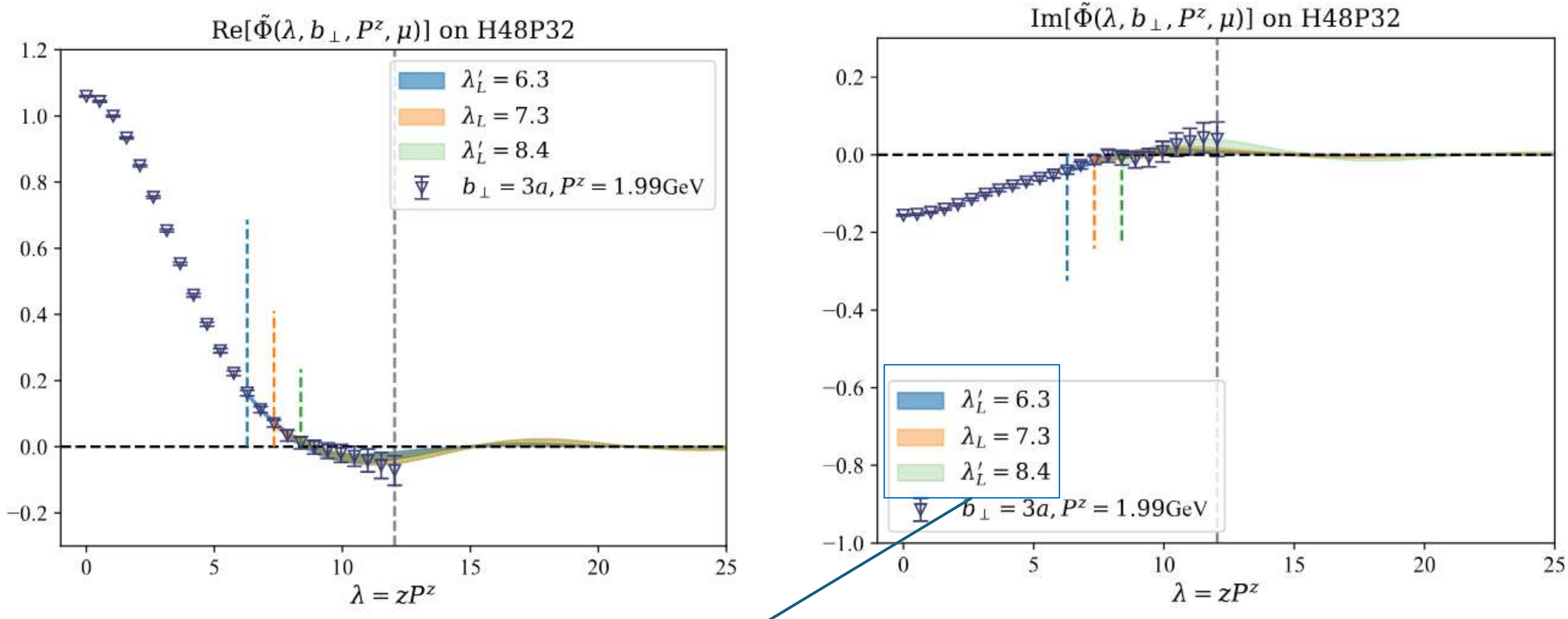
$$\tilde{\Psi}^{\pm}(x, b_\perp, \mu, \zeta_z) = \lim_{L\to\infty} \int \frac{dzP^z}{2\pi} e^{ixzP^z} \frac{\tilde{\Phi}^{\pm 0}(z, b_\perp, P^z, L)}{Z_O(\mu, a)\sqrt{Z_E(2L + z, b_\perp, \mu)}},$$

$Z_E =$



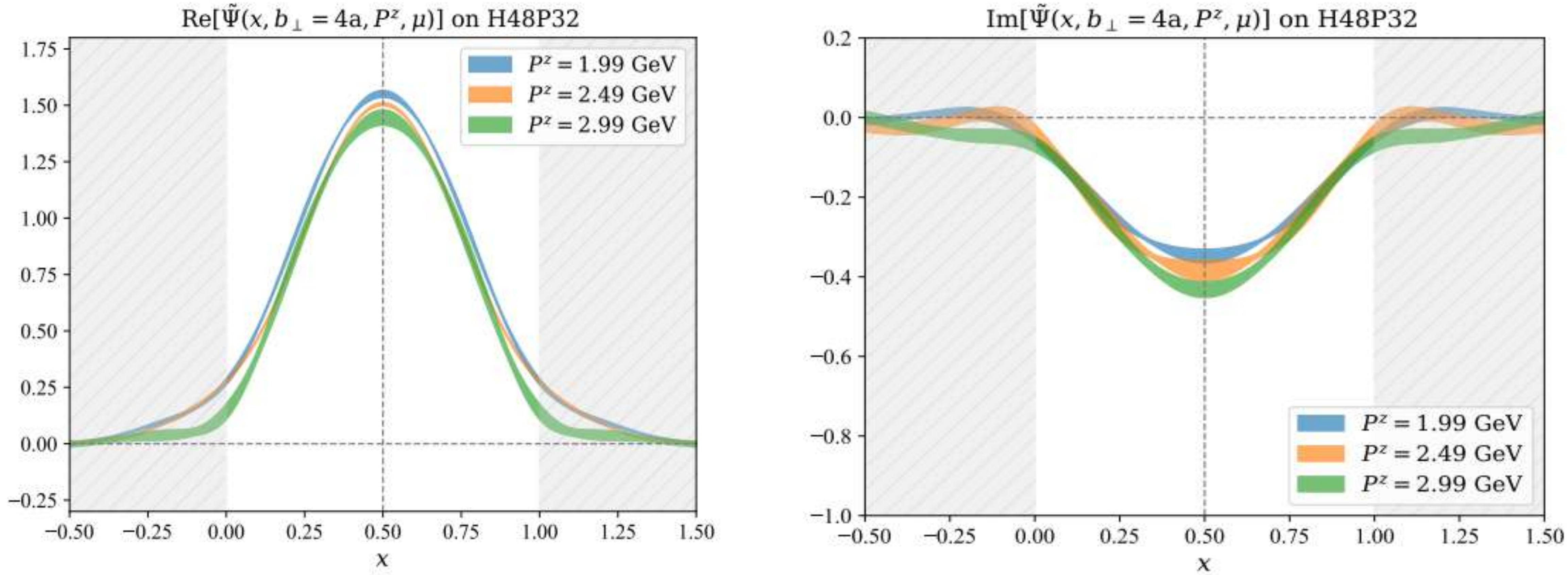
Large-λ extrapolation and Fourier Transformation

$$\lim_{\lambda\to\infty}\tilde{\Phi}\left(z,b_\perp,P^z,a,L\right)=[f_1(b_\perp)+if_2(b_\perp)][\frac{c_1}{(-i\lambda)}+e^{i\lambda}\frac{c_2}{(i\lambda)^d}]e^{\frac{\lambda}{\lambda_0}}$$



Take different selections of λ as systematic uncertainty.

Large-λ extrapolation and Fourier Transformation



P^z dependence of quasi-TMDWFs is the key to extract CS kernel!

Quasi-TMDWF

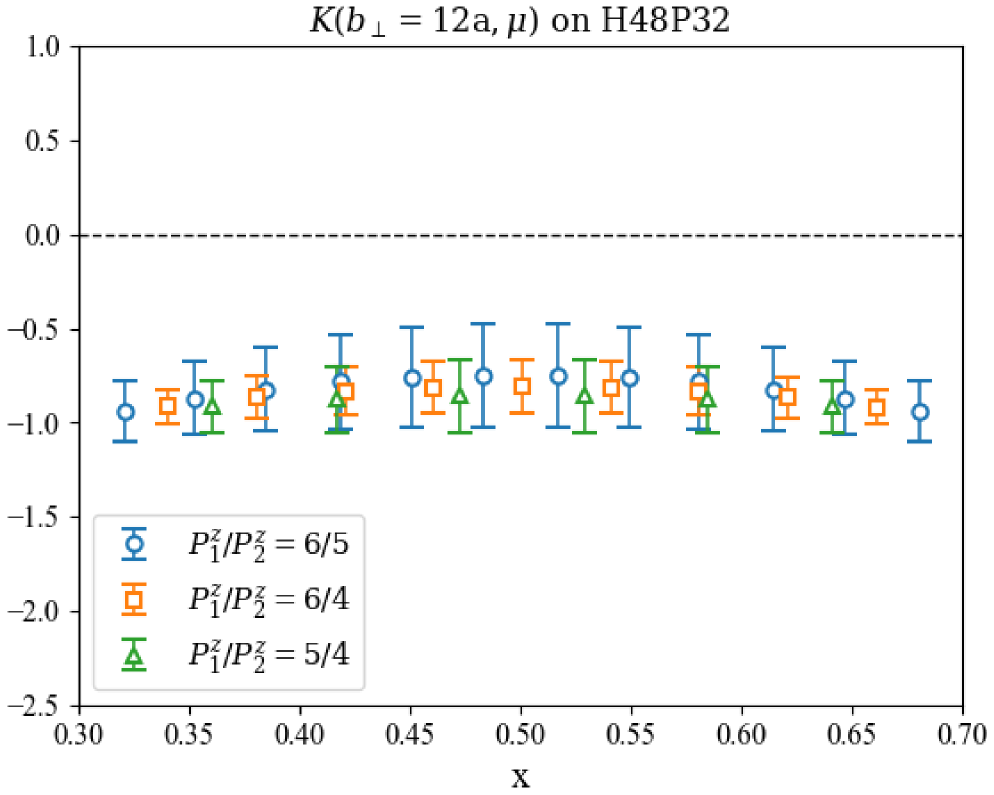
Collins-Soper Kernel

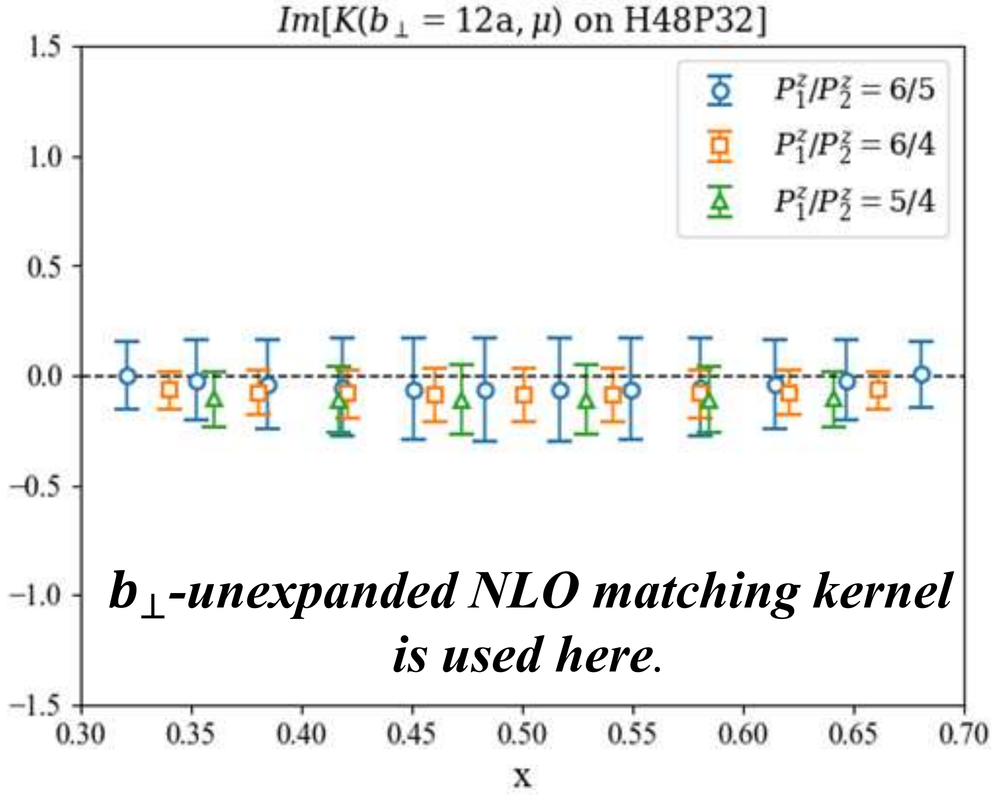
$$K(b_\perp, \mu) = \frac{1}{\ln\left(P_1^z / P_2^z\right)} \ln\left[\frac{H_\Gamma\left(\zeta_2^z, \mu\right)\, \tilde{\Phi}\left(x, b_\perp, \zeta_1^z, \mu\right)}{H_\Gamma\left(\zeta_1^z, \mu\right)\, \tilde{\Phi}\left(x, b_\perp, \zeta_2^z, \mu\right)}\right]$$

Intrinsic Soft Function

CS Kernel Extraction

$$K(b_\perp,\mu)=\frac{1}{\ln(P_1^z/P_2^z)}\ln\left[\frac{H_\Gamma(\zeta_2^z,\mu)\,\tilde{\Phi}(x,b_\perp,\zeta_1^z,\mu)}{H_\Gamma(\zeta_1^z,\mu)\,\tilde{\Phi}(x,b_\perp,\zeta_2^z,\mu)}\right]$$





$b_\perp$-unexpanded NLO matching kernel is used here.

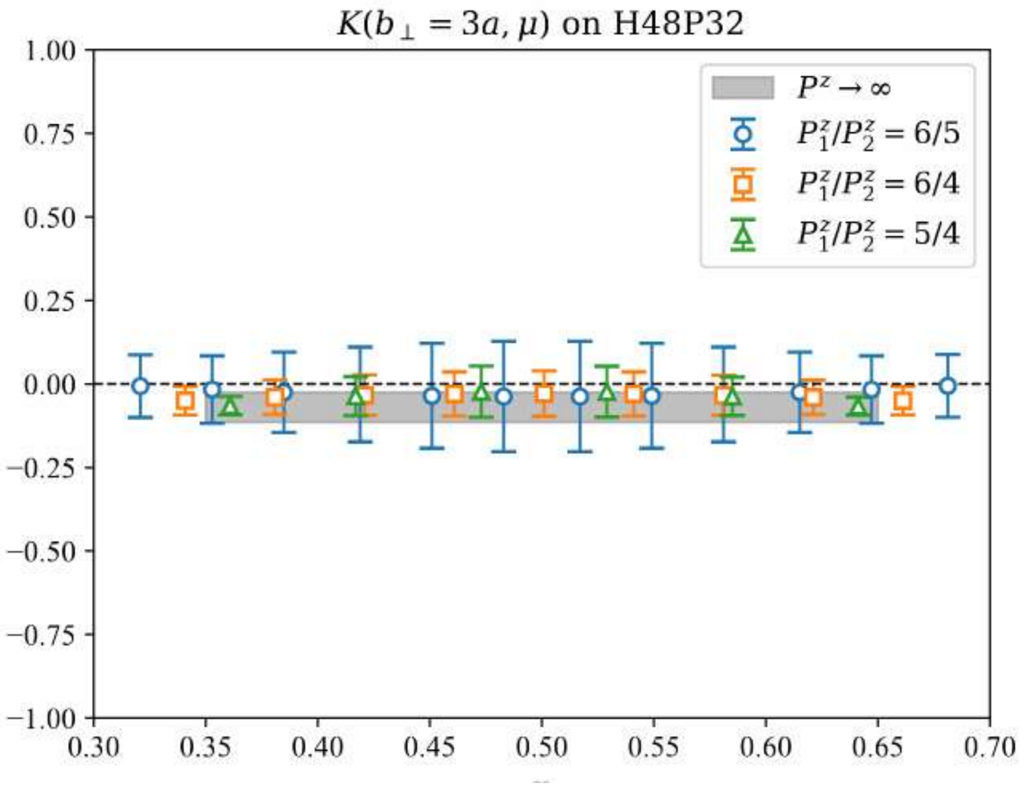
(For better visualization, we present a subset of points selected from the 200 data points in each of the 3 cases.)

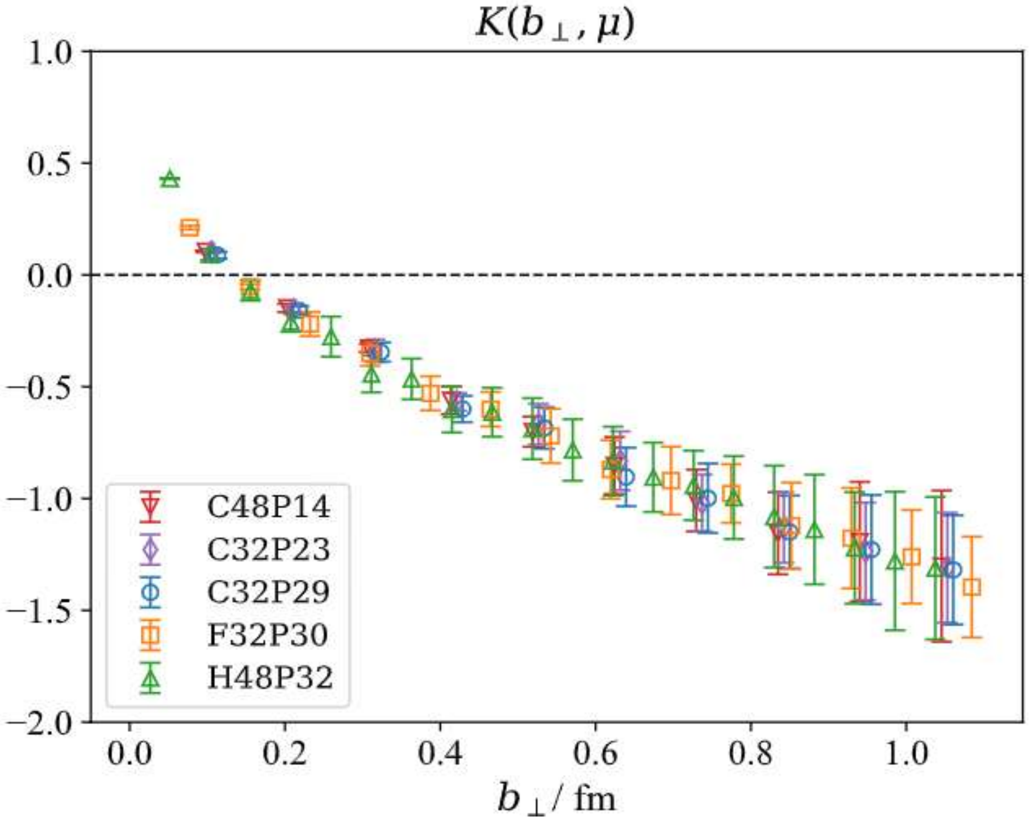
For different momentum ratios, the results keep stable.

Tan et al.(LPC), Phys.Rev.D 113 (2026); Avkhadiev et al., Phys.Rev.D 108 (2023)

Large Momentum Extrapolation

$$K(b_\perp, \mu; x, P_1^z, P_2^z, a, m_\pi) = K(b_\perp, \mu; a, m_\pi) + A(b_\perp, \mu; x, a, m_\pi)\left[\frac{1}{(P_1^z)^2} - \frac{1}{(P_2^z)^2}\right]$$





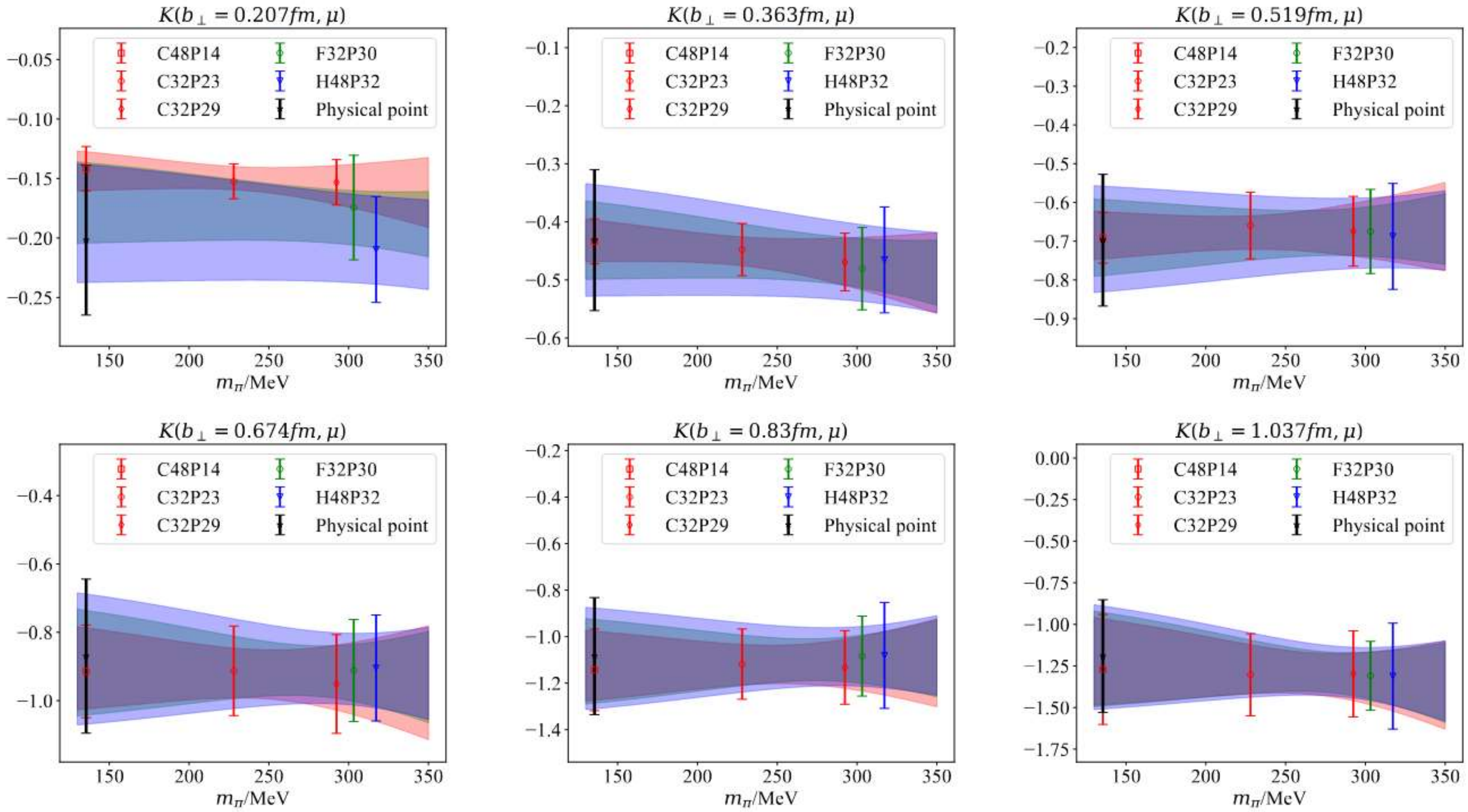
The result in the large momentum limit is stable.

Results on different ensembles are highly consistent!

Tan et al.(LPC), Phys.Rev.D 113 (2026)

Continuum and Physical Pion Mass Extrapolation

$$K(b_\perp,\mu;a,m_\pi) = K(b_\perp,\mu) + a^2 B(b_\perp,\mu) + (m_\pi^2 - m_{\pi,phy}^2)C(b_\perp,\mu)$$

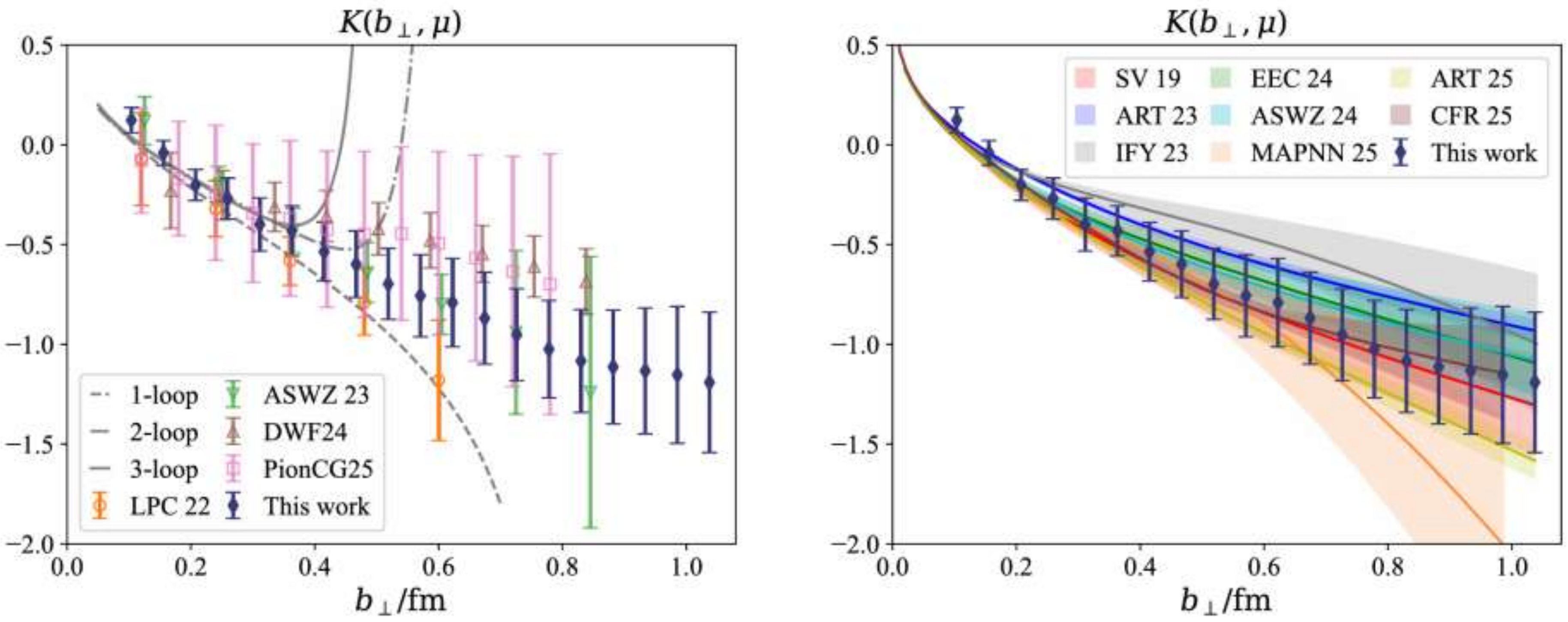


Short-distance region is more sensitive to discretization.

Final CS kernel are obtained after extrapolation.

Tan et al.(LPC), Phys.Rev.D 113 (2026)

Results of CS Kernel



- ✓ **Continuum limit;**
- ✓ **Physical mass;**
- ✓ $\boldsymbol{b_\perp = 1\text{fm}}$

Lattice:

Chu et al. (LPC), JHEP 08, 172 (2023);
Avkhadiev, et al., PRD 108, 114505 (2023);
Bollweg, et al., PRB 852, 138617 (2024);
Avkhadiev et al., PRL. 132, 231901 (2024);
Bollweg, et al., PRD 112, 034501(2025).

Phenomenological:

Scimemi et al., JHEP 06, 137 (2020);
Moos et al., JHEP 05, 036 (2024);
Isaacson et al., PRD 110, 073002 (2024);
Kang et al., arXiv:2410.21435;
Bacchetta et al., PRL. 135, 021904 (2025);
Moos et al., arXiv:2503.11201;
Camarda et al., arXiv:2508.06201.

Collins-Soper Kernel

Quasi-TMDWF

Intrinsic Soft Function

$$S_I(b_\perp,\mu) = \frac{F(b_\perp,P_1,P_2,\mu)}{\int \mathrm{d}x_1 \mathrm{d}x_2 H(x_1,x_2)\tilde{\Phi}^\dagger(x_2,\zeta_2^z,b_\perp,\mu)\tilde{\Phi}(x_1,\zeta_1^z,b_\perp,\mu)}$$

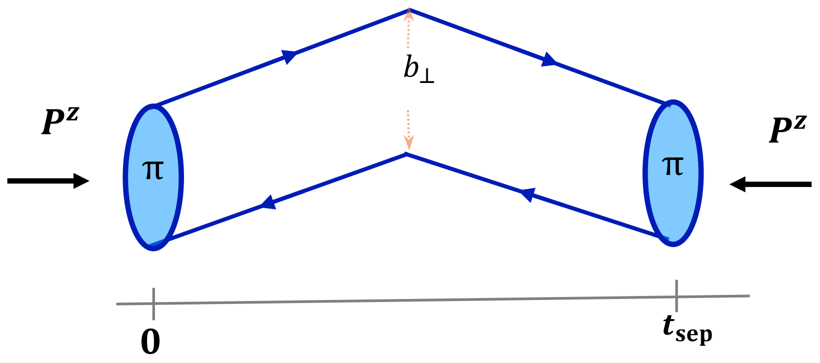
Four Quark Form Factor

- **From the Form Factor Factorization:**

$$S_I(b_\perp,\mu)=\frac{F(b_\perp,P_1,P_2,\mu)}{\int \mathrm{d}x_1\mathrm{d}x_2 H(x_1,x_2)\tilde{\Phi}^\dagger(x_2,\zeta_2^z,b_\perp,\mu)\tilde{\Phi}(x_1,\zeta_1^z,b_\perp,\mu)}$$

- **Four-quark form factor:**

$$F(b_\perp,P_1,P_2,\mu,\Gamma)=\frac{\langle P_2|\bar{q}(b_\perp)\Gamma q(b_\perp)\bar{q}(0)\Gamma q(0)|P_1\rangle}{\langle 0|\bar{q}(0)\gamma^\mu\gamma^5 q(0)|P_1\rangle\langle P_2|\bar{q}(0)\gamma_\mu\gamma^5 q(0)|0\rangle}$$



Ji et al. , Nucl.Phys.B 955 (2020)

Correlator & Fit

$$\frac{C_3(b_\perp, \Gamma, t_{\rm seq}, t, P^z)}{2\left[C_2(0,0,0,t_{\rm seq}/2, P^z)\right]^2} = F(b_\perp, \Gamma, P^z)\frac{1 + c_1(e^{-\Delta E t} + e^{-\Delta E(t_{\rm seq}-t)})}{1 + c_2 e^{-\Delta E t_{\rm seq}/2}}$$

- **Leading twist:**

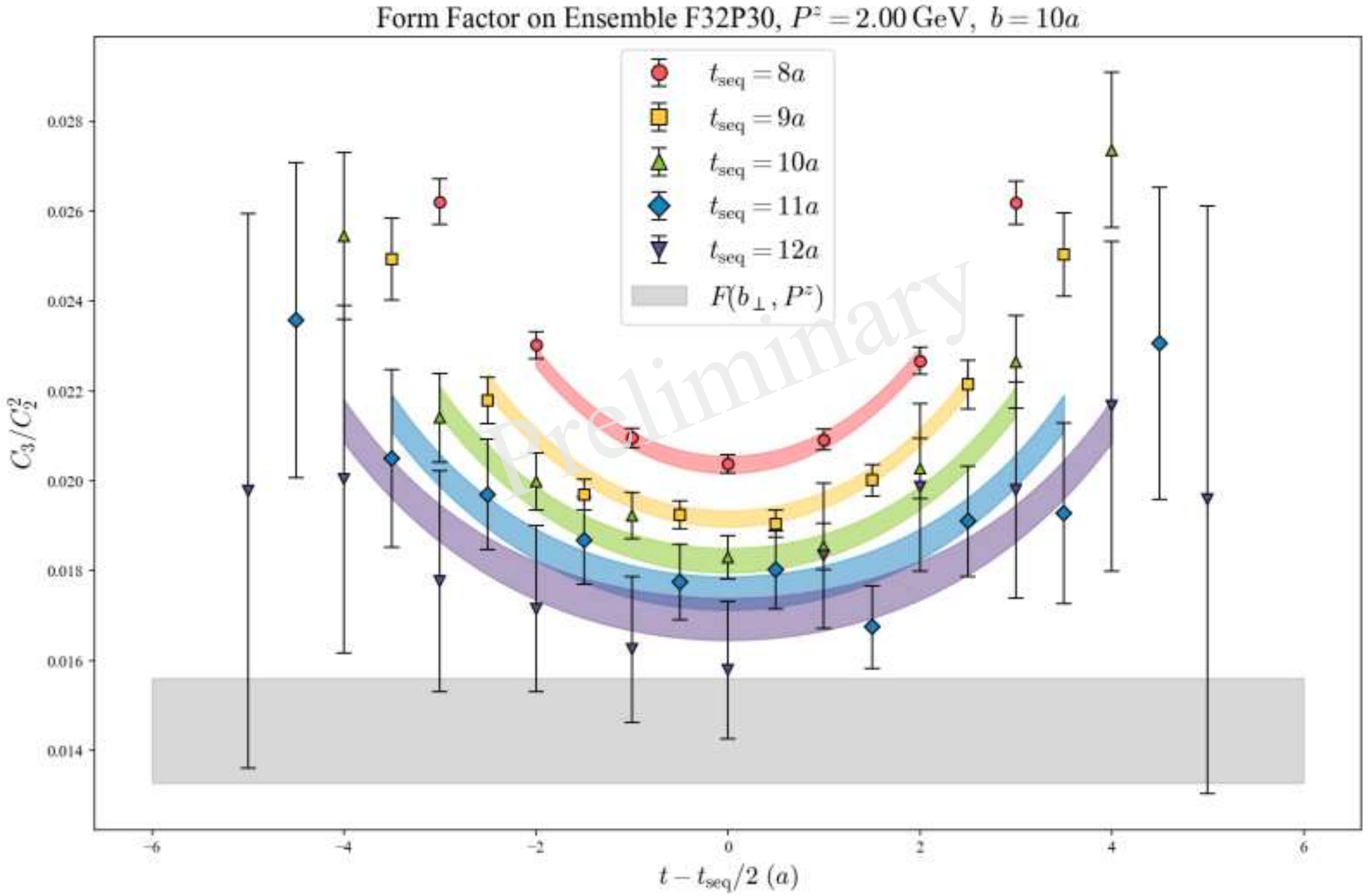
$$F(\Gamma = \gamma_\perp) + F(\Gamma = \gamma_\perp\gamma_5)$$

Deng et al., JHEP 09 (2022).

- **Kinetically enhanced interpolator:**

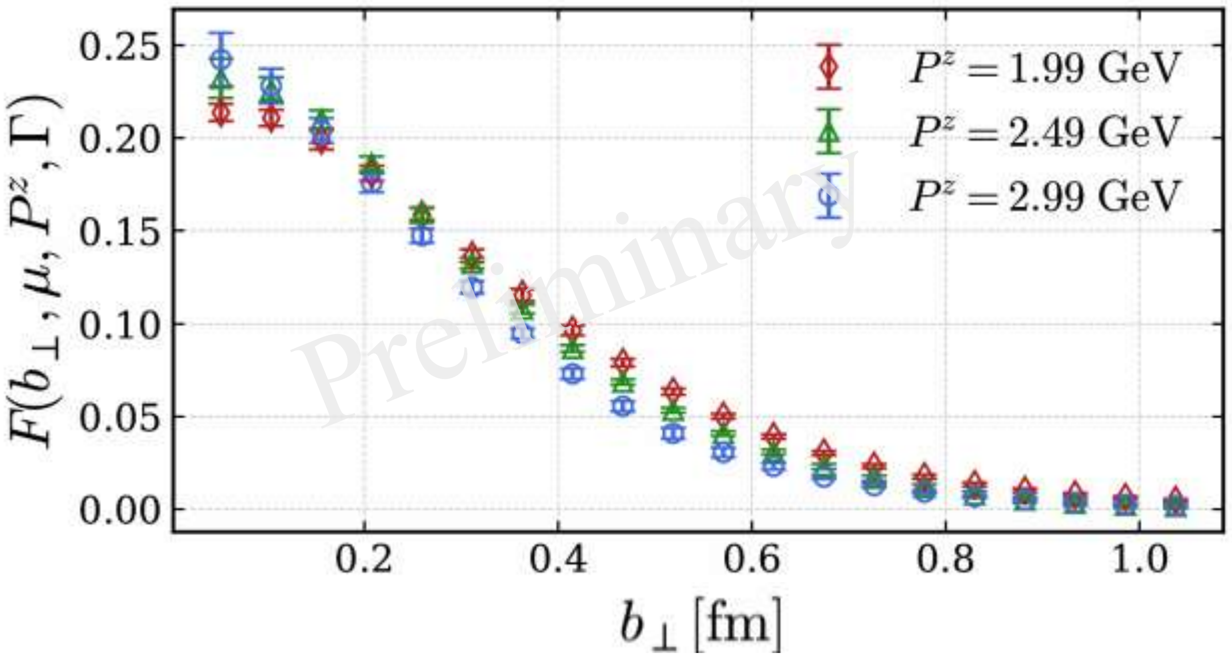
$$\bar{u}\gamma_5 d \rightarrow \bar{u}\gamma_\mu\gamma_5 d$$

Zhang et al., Phys.Rev.D 112 (2025)

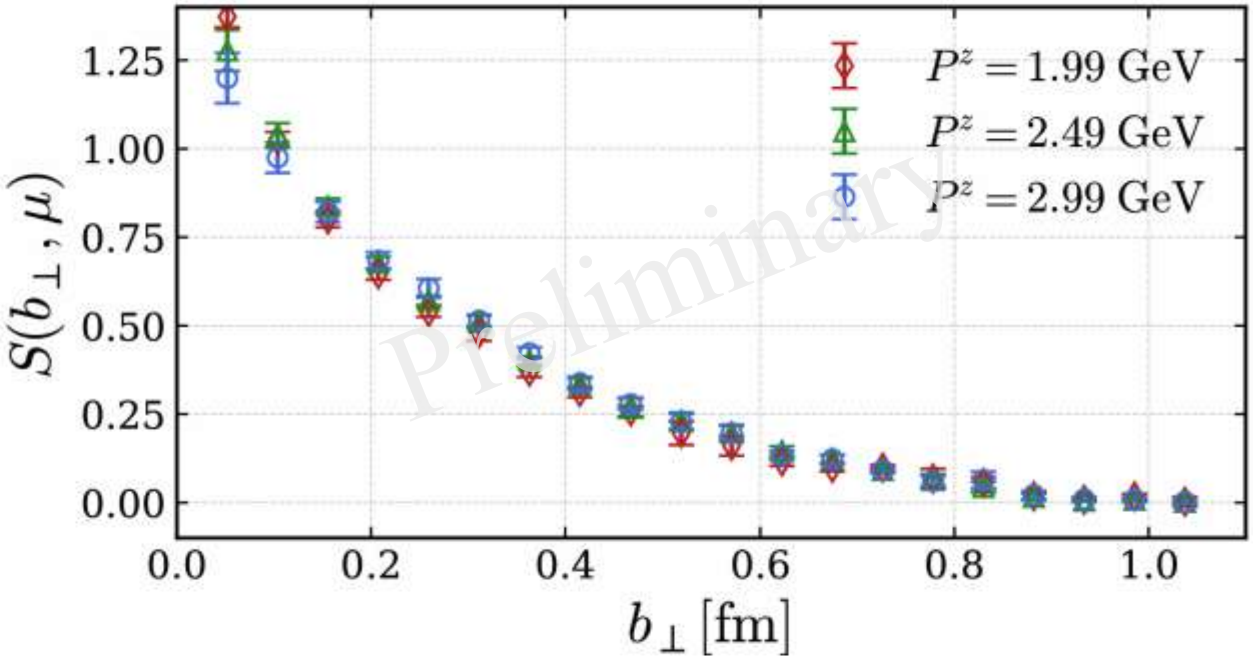


From Form Factor to Intrinsic Soft Function

Form Factor —— ***NLO matching + quasi-TMDWFs*** ——> S_I



Form factor on ensemble H48P32.

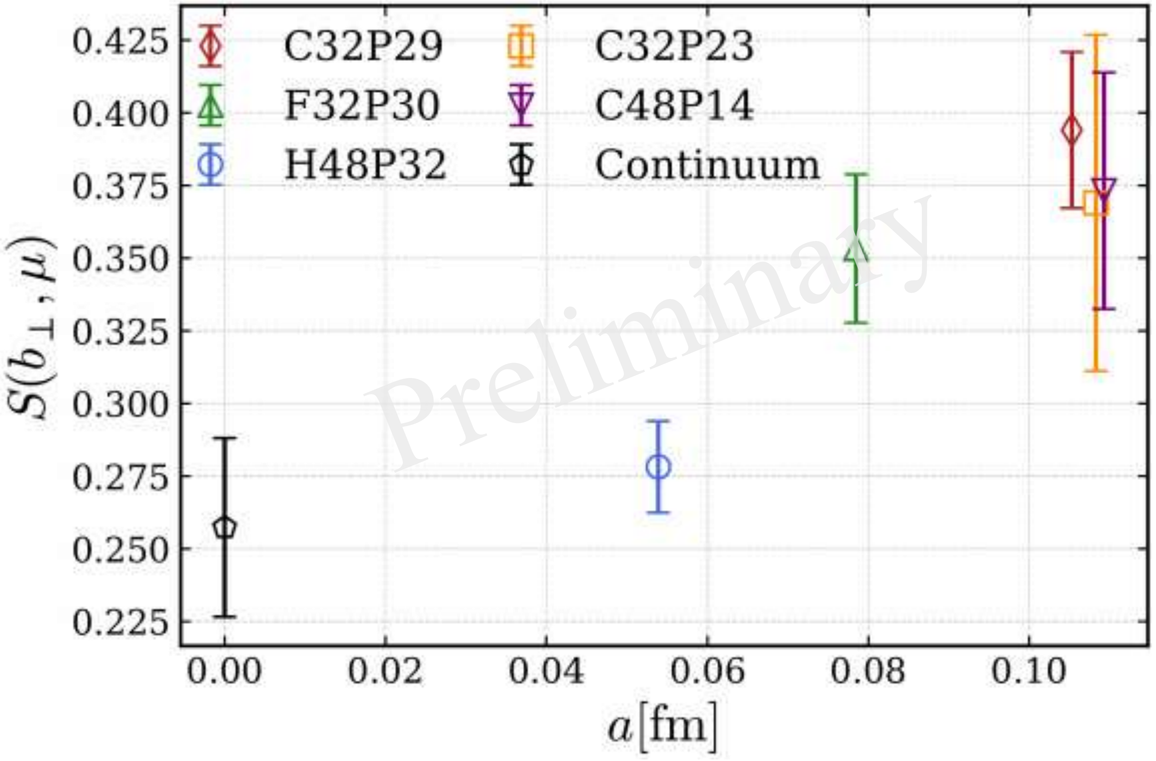


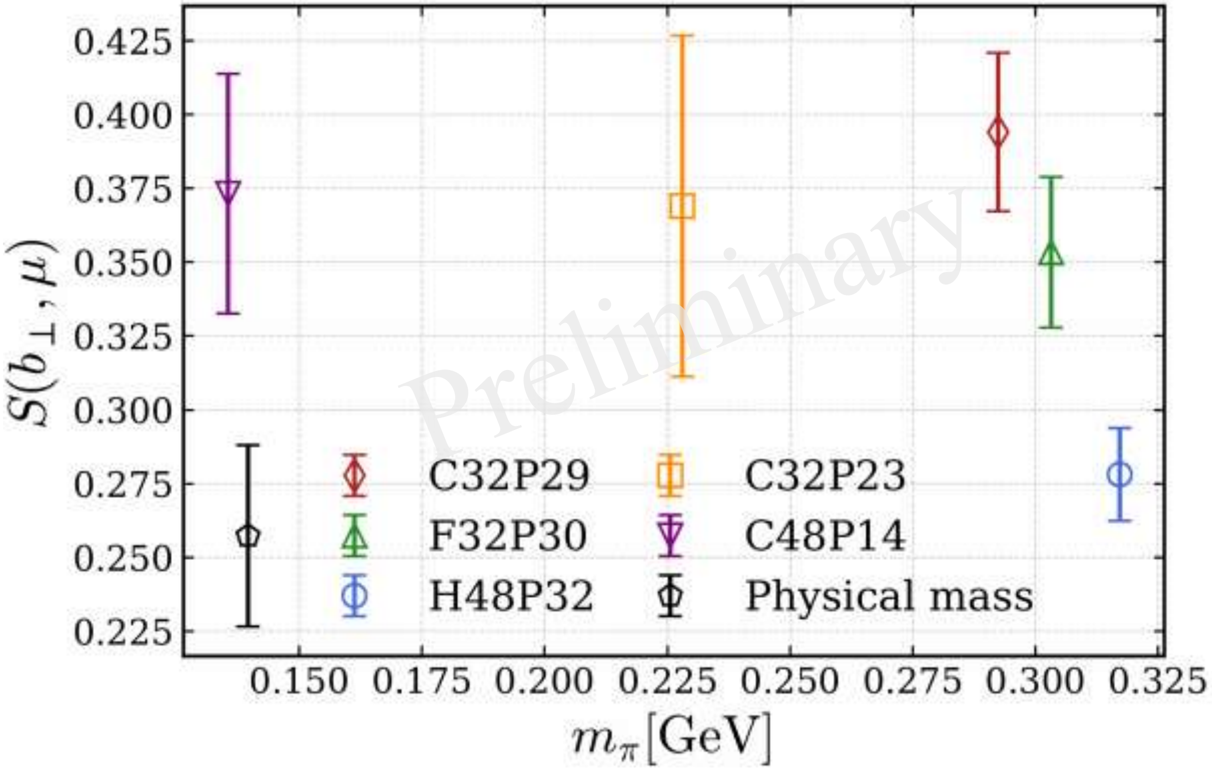
Intrinsic soft function on ensemble H48P32.

Ji et.al., Nucl.Phys.B 955 (2020),
Deng et al., JHEP 09 (2022).

Extrapolations

$$S_I(b_\perp,\mu) = S_{\lim}(b_\perp,\mu) + \frac{A}{(P^z)^2} + Ba^2 + C(m_{\text{latt}}^2 - m_{\text{phy}}^2)$$





S_I with different pion masses are consistent with each other.

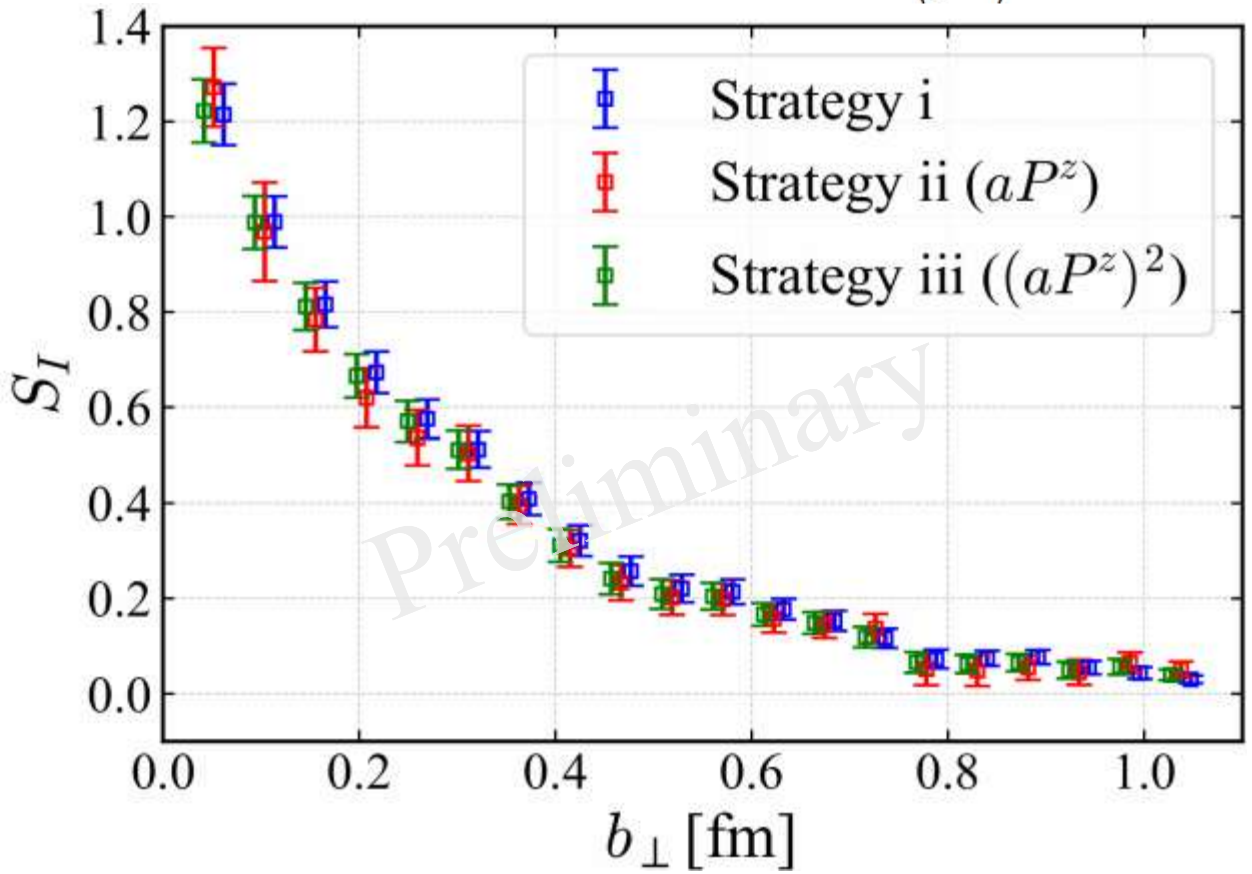
Li et al., Phys.Rev.Lett. 128 (2022)

Extrapolations

Strategy i $$S_I(b_\perp,\mu) = S_{\text{lim}}(b_\perp,\mu) + \frac{A}{(P^z)^2} + Ba^2 + C(m_{\text{latt}}^2 - m_{\text{phy}}^2)$$

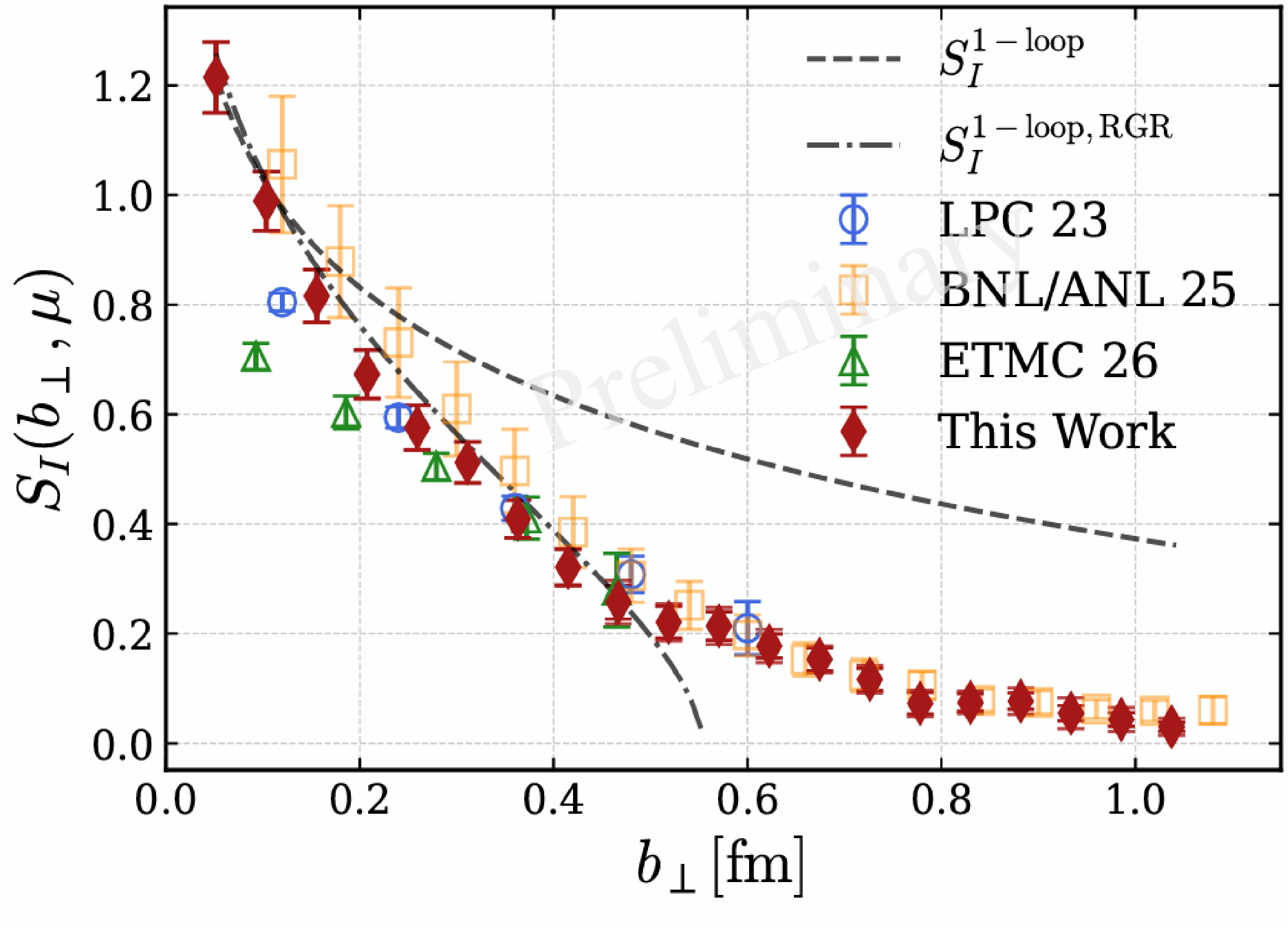
Strategy ii $$S_I(b_\perp,\mu) = S_{\text{lim}}(b_\perp,\mu) + \frac{A}{(P^z)^2} + Ba^2 + C(m_{\text{latt}}^2 - m_{\text{phy}}^2) + DaP^z$$

Strategy iii $$S_I(b_\perp,\mu) = S_{\text{lim}}(b_\perp,\mu) + \frac{A}{(P^z)^2} + Ba^2 + C(m_{\text{latt}}^2 - m_{\text{phy}}^2) + D(aP^z)^2$$



The cross terms hardly affect the final results of S_I.

Result of Intrinsic Soft Function



- ✓ **Continuum limit;**
- ✓ **Physical mass;**
- ✓ $\boldsymbol{b_\perp = 1\mathrm{fm}}$

Chu et al.(LPC), JHEP 08 (2023);
Bollweg et al., Phys.Rev.D 112 (2025);
Alexandrou et al., Phys.Rev.D 113 (2026)

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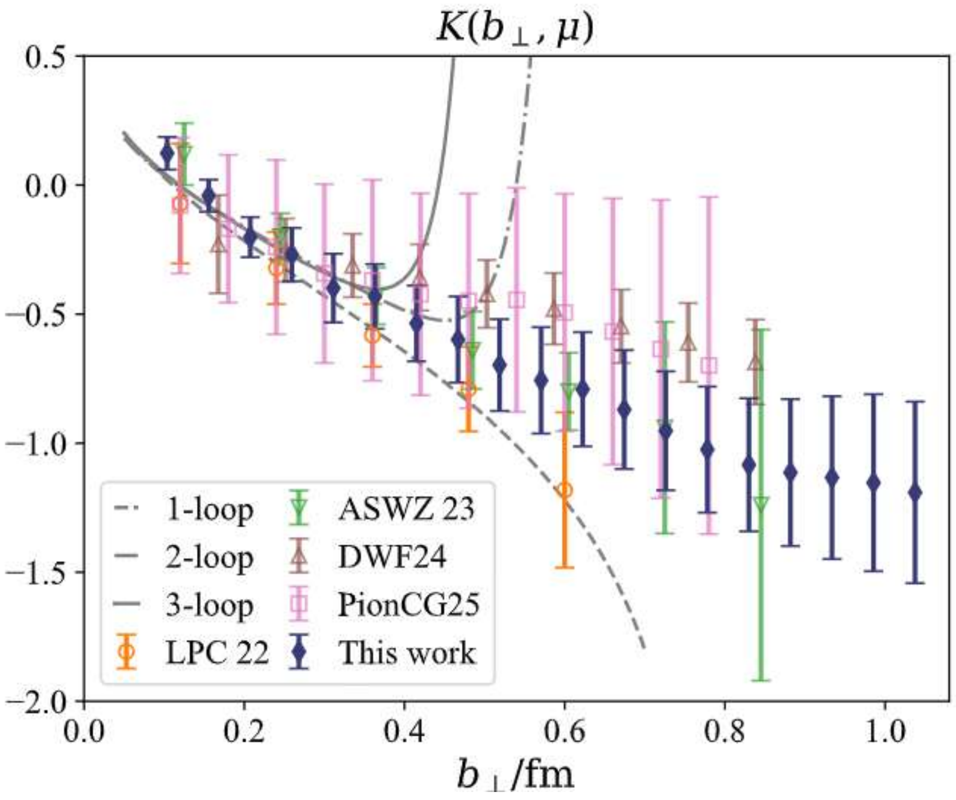
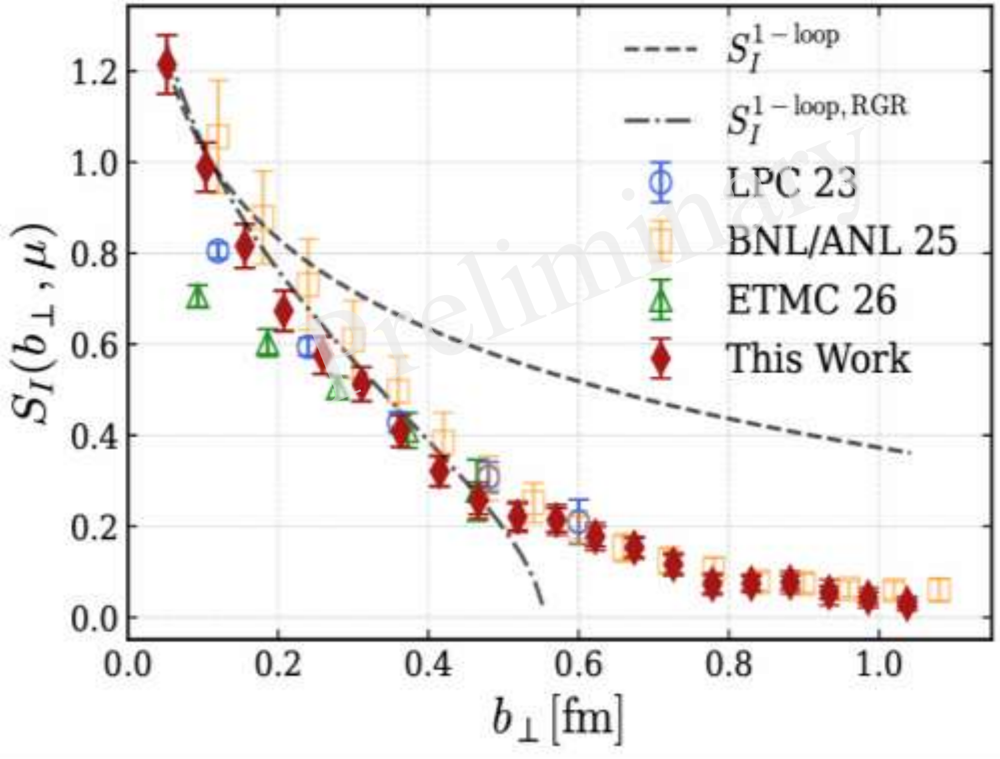
Summary & Outlook

- Collins-Soper kernel and the intrinsic soft function are essential nonperturbative inputs for TMDs;
- Both results are extrapolated to the continuum limit and the physical pion mass, and reach $b_\perp \sim 1\text{fm}$ with high accuracy.



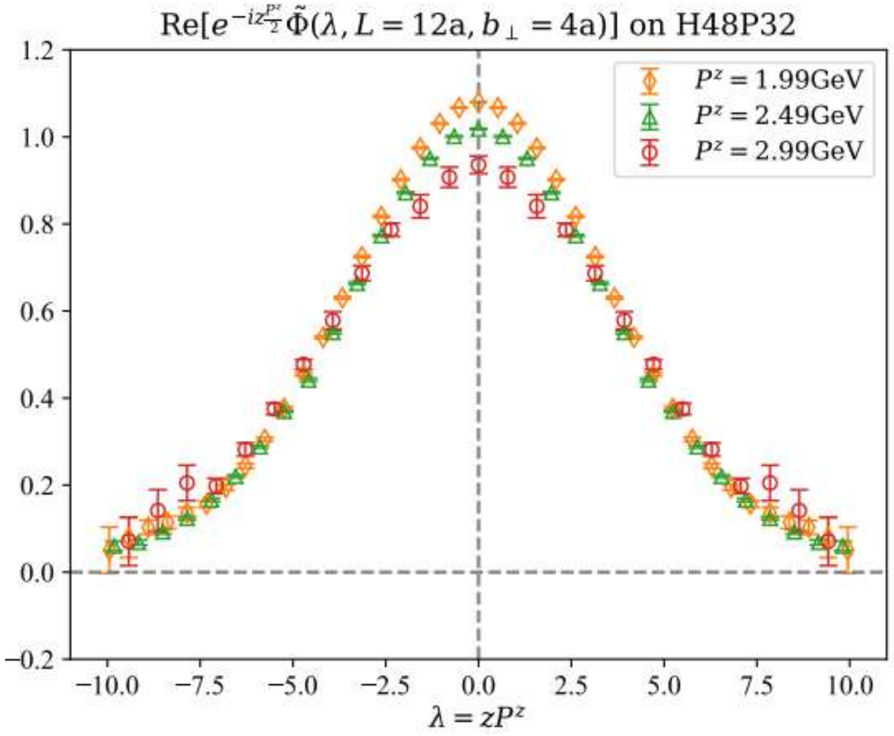


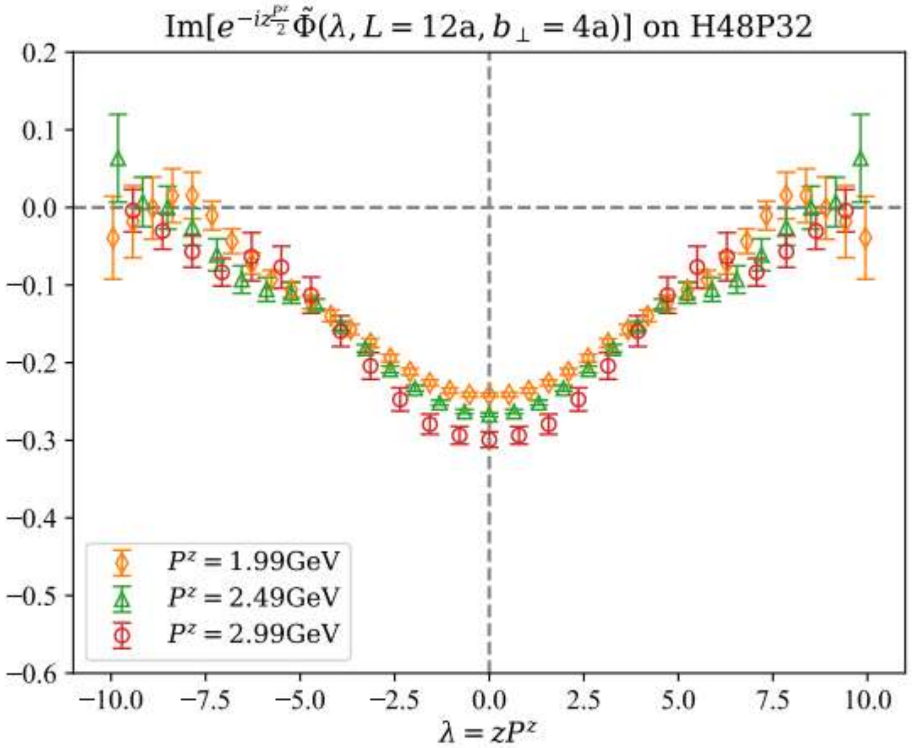
Thank you!

Back Up

Subtracted Quasi-TMDWF

$$\tilde{\Phi}^{\pm}(z, P^z, L, b, \mu) = \frac{\tilde{\Phi}^{\pm 0}(z, b_\perp, P^z, a, L)}{Z_O(\mu, a)\sqrt{Z_E(2L + z, b_\perp, \mu, a)}}$$

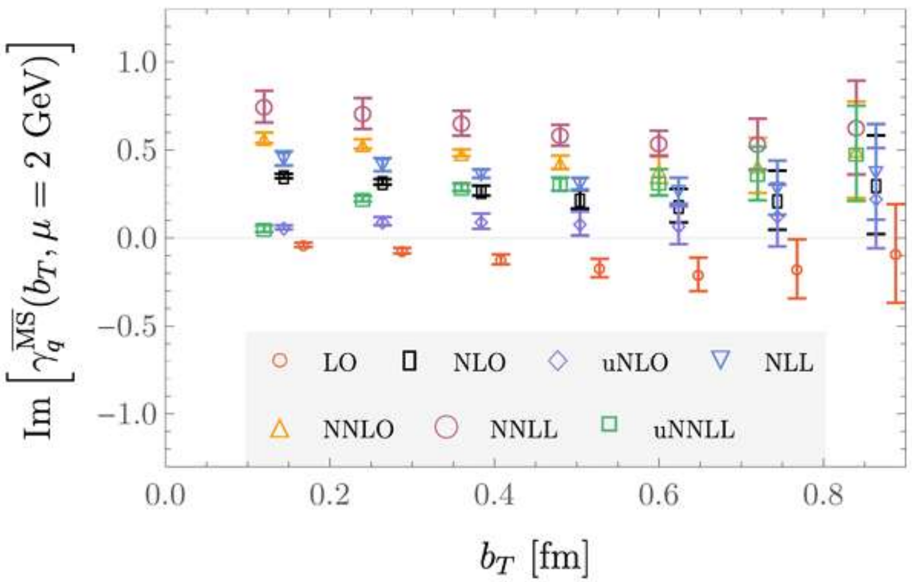




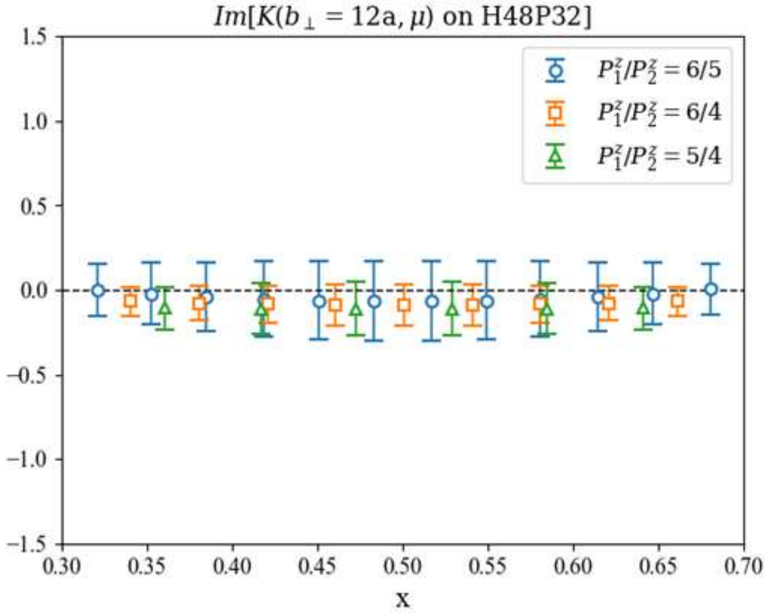
Different results have similar behavior.

Residual imaginary part

- ➢ The CS kernel has an unphysical imaginary part under NLO matching, which is treated as systematic uncertainty of CS kernel: $\sigma_{\text{sys}}^{\text{Im}} = \sqrt{K(b_\perp,\mu)^2 + \text{Im}[K(b_\perp,\mu)^2]} - |K(b_\perp,\mu)|$
- ➢ Inspired by *Phys.Rev.D* 108 (2023) 11, we consider the $b_\perp$-unexpanded NLO matching kernel to reduce the imaginary part of CS kernel.



Avkhadiev et al., Phys.Rev.D 108 (2023)



Tan et al.(LPC), Phys.Rev.D 113 (2026)

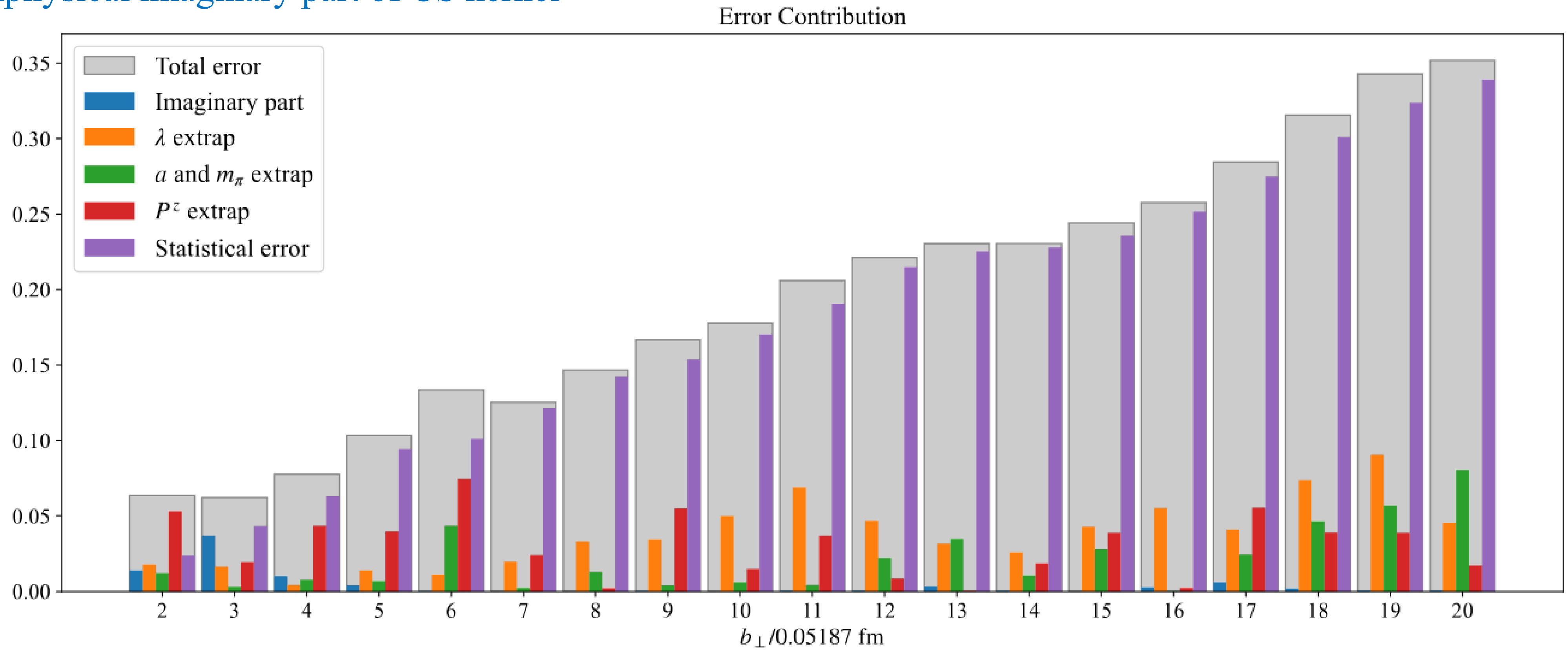
Uncertainty Analysis

	CS Kernel	Intrinsic Soft Function
Common	$\delta^{P^z, m_\pi, a}$: The difference between extrapolated result and original data with closest momentum, pion mass or lattice spacing; δ^{λ}: The difference between different windows in large λ extrapolation of quasi-TMDWF. δ^{stat}: The statistical uncertainty.	
Different	δ^{Im}:The residual imaginary part of CS kernel.	/

Systematic Uncertainty: CS Kernel

$\delta^{P^z,m_\pi,a}$： uncertainty in large momentum, continuum and physical pion mass limit extrapolation;
δ^{λ}： uncertainty in the large-λ extrapolation of quasi-TMDWFs

δ^{Im}: unphysical imaginary part of CS kernel



Systematic Uncertainty: Intrinsic Soft Function

$\delta^{P^z, m_\pi, a}$： uncertainty in large momentum, continuum and physical pion mass limit extrapolation;

δ^{λ}： uncertainty in the large-λ extrapolation of quasi-TMDWFs