Non-equilibrium criticality in the synchronization of lattices of self-sustained oscillators

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SYNCHRONIZATION

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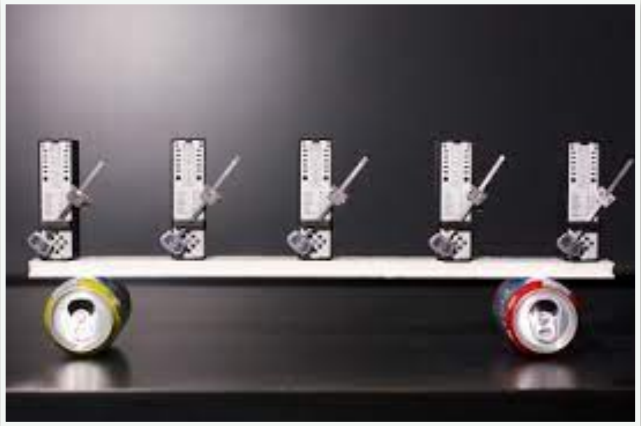

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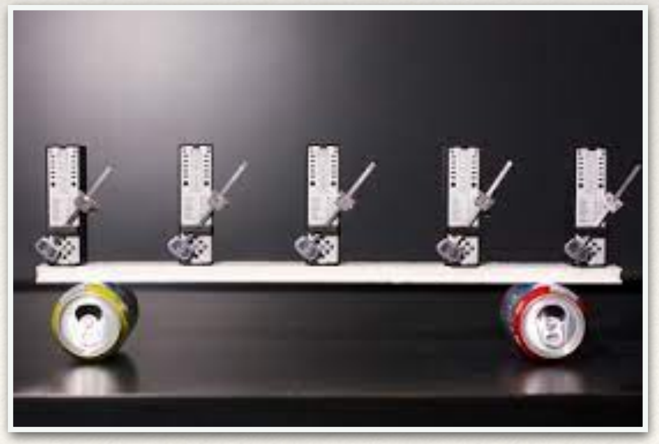
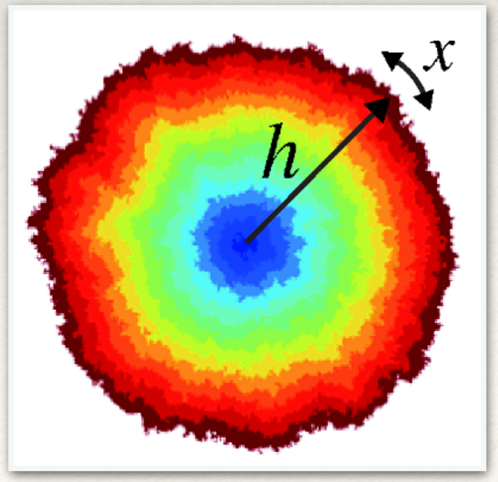


SURFACE KINETIC ROUGHENING

- Universality in growth processes and other non-equilibrium phenomena.
- Examples: Coffee-ring formation, growth of bacterial colonies, ice-flake deposition, thin-film production…
- Models: Discrete growth models and interfacial continuum equations, with thermal or quenched disorder.



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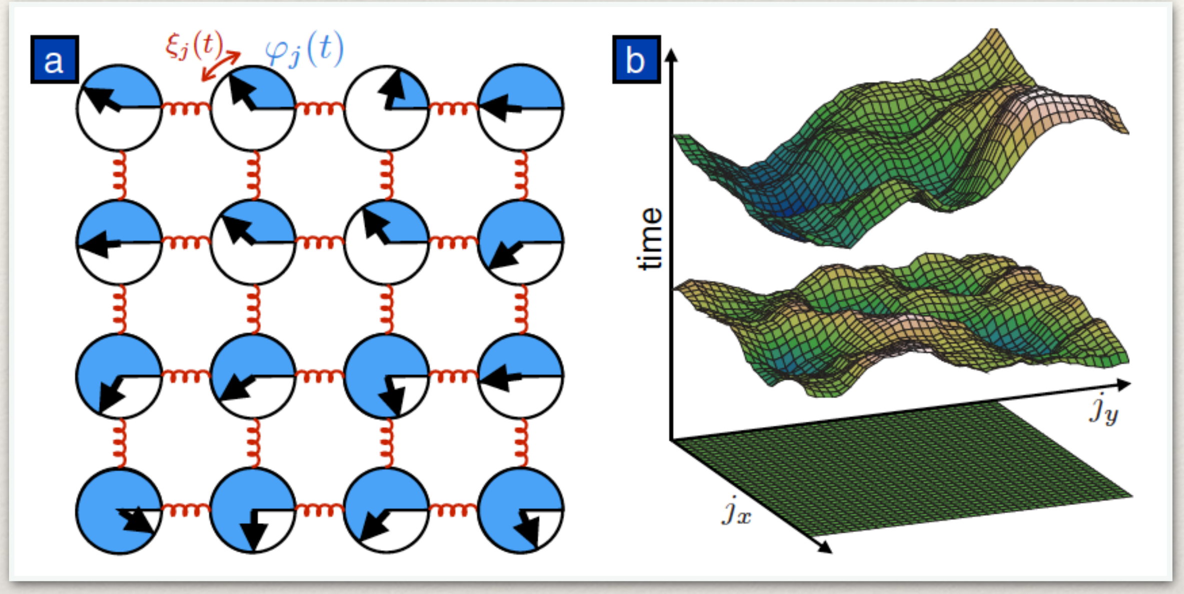

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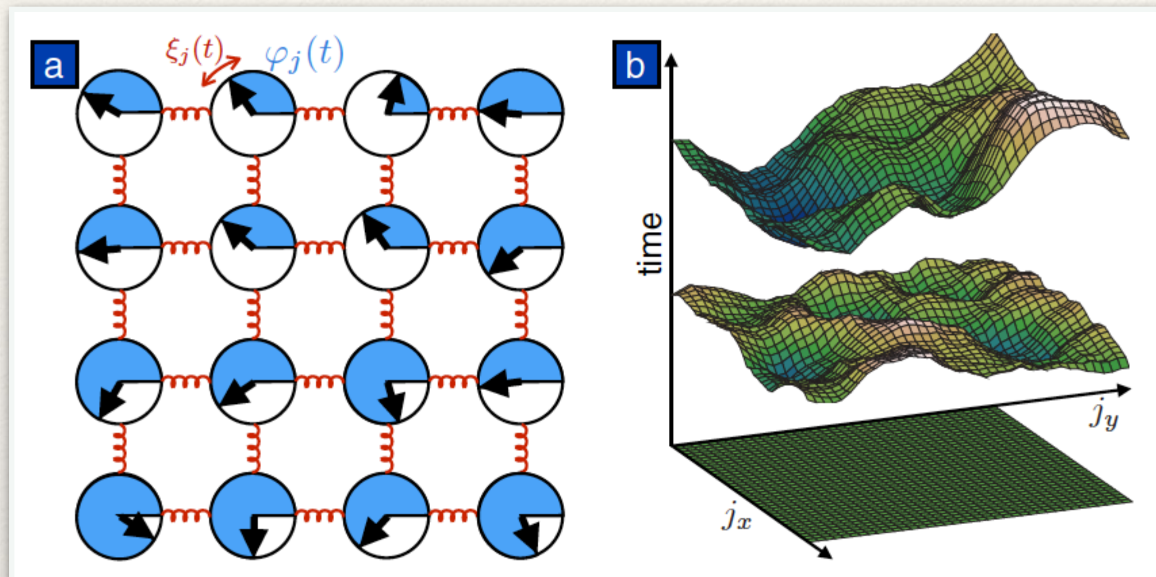

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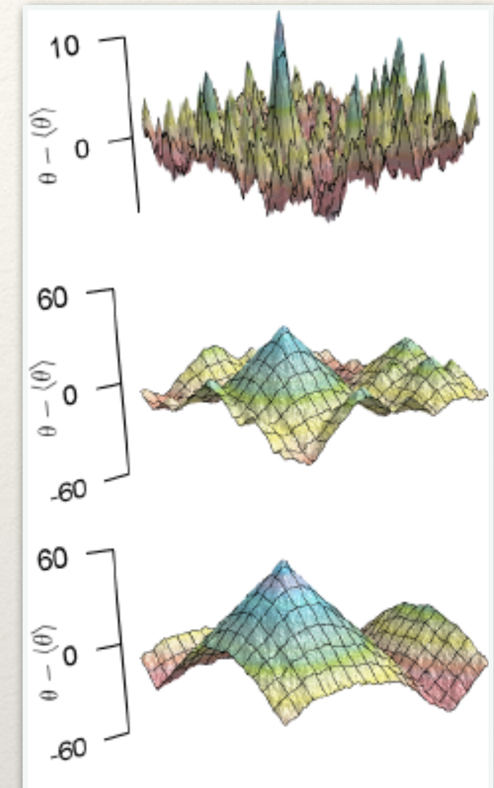

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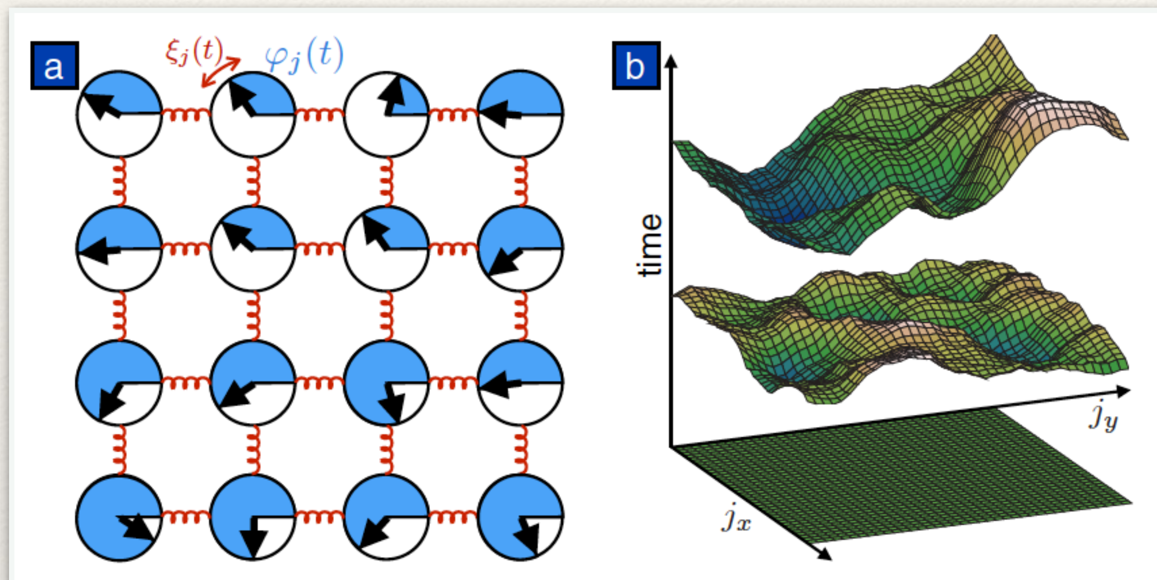

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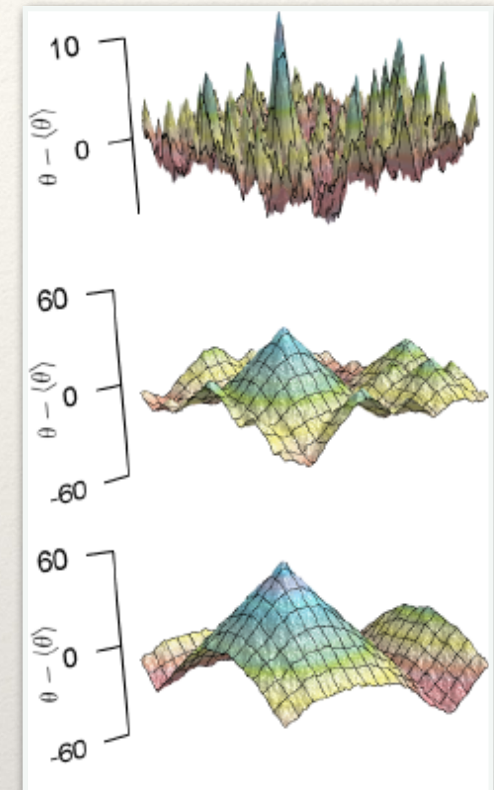

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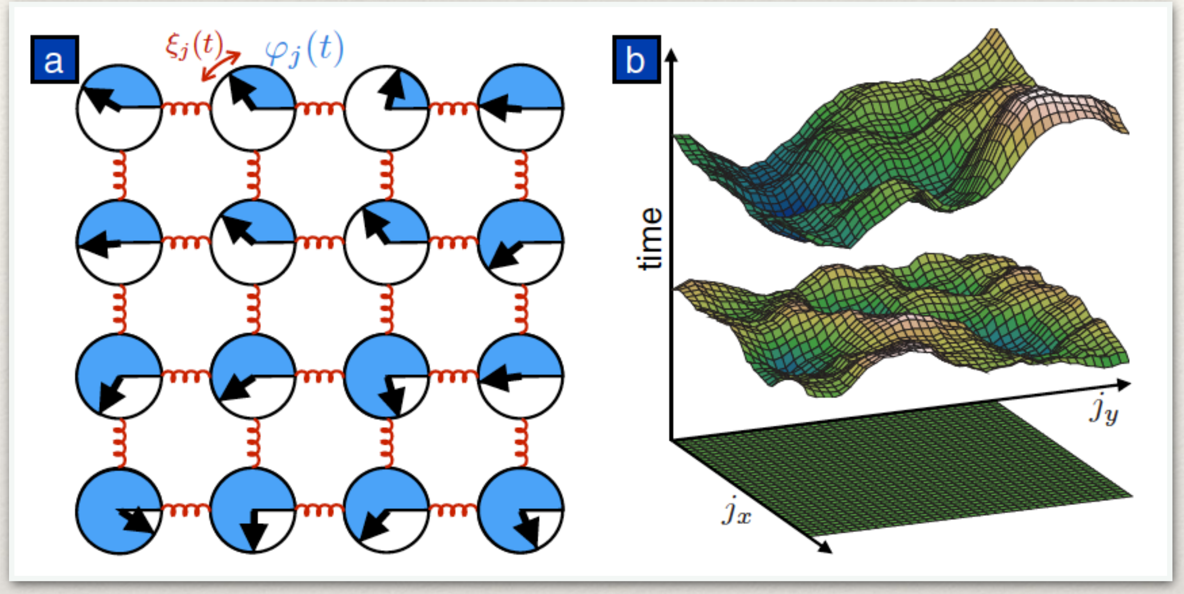

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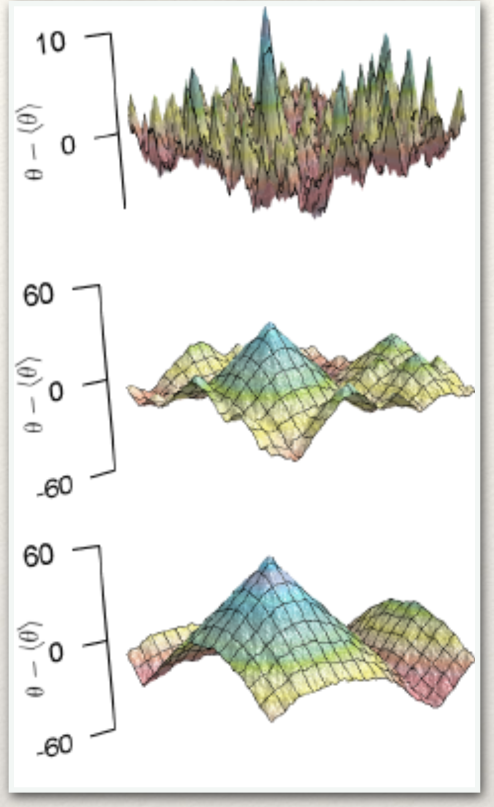

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- Analogy recently explored in some specific contexts, including bosonic systems and routes out of synchronization.

- Mathematical connection between main models of phase oscillators and interfaces (see, e. g., A. Pikovsky, M. Rosenblum and J. Kurths, *Synchronization*, CUP, 2001).

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- The synchronization process is endowed with universal features recently studied in the physics of surface kinetic roughening.
- In fact, both synchronization and kinetic roughening are instances of non-equilibrium criticality.

Some background:
synchronization of coupled oscillators

Kuramoto model

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- Long-time dynamics of weakly coupled, nearly identical limit-cycle oscillators

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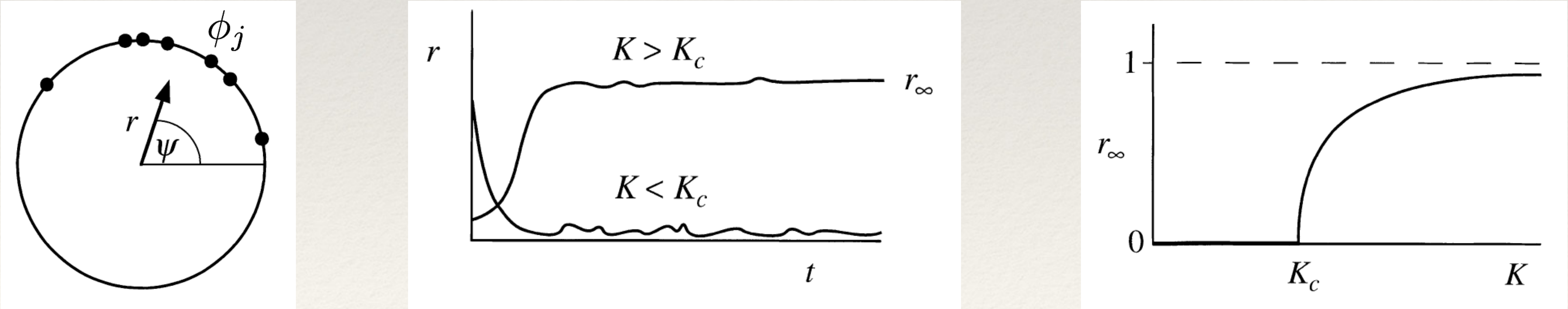

S. H. Strogatz, Physica D 143, 1 (2000)

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    - For $\delta \neq 0$ finite K^* for all L, attributed to the lack of odd symmetry in Γ (Ostborn, Phys. Rev. E 70, 016120, 2004).

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surface kinetic roughening

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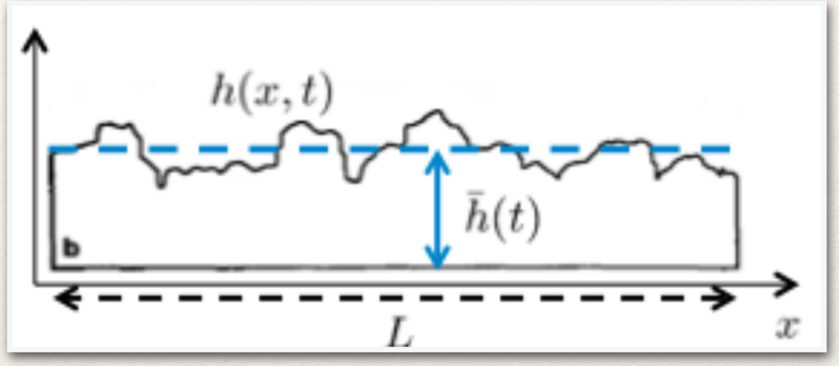
- Dynamics of height $h(\mathbf{x}, t)$ over substrate position $\mathbf{x}$. Dominant terms preserving reasonable symmetries

$$\partial_t h = \nu \nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta$$



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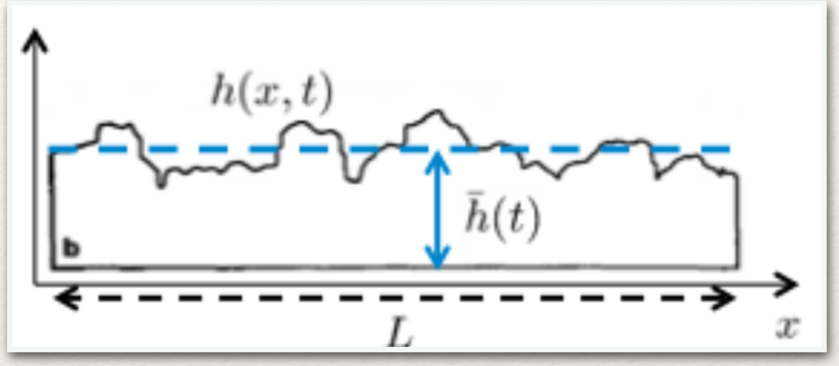

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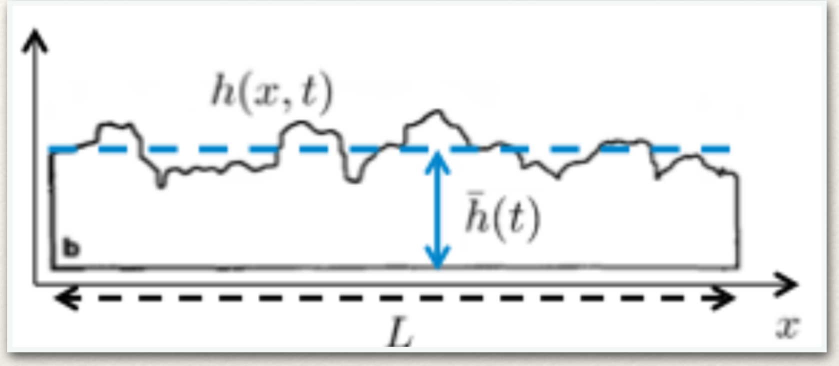

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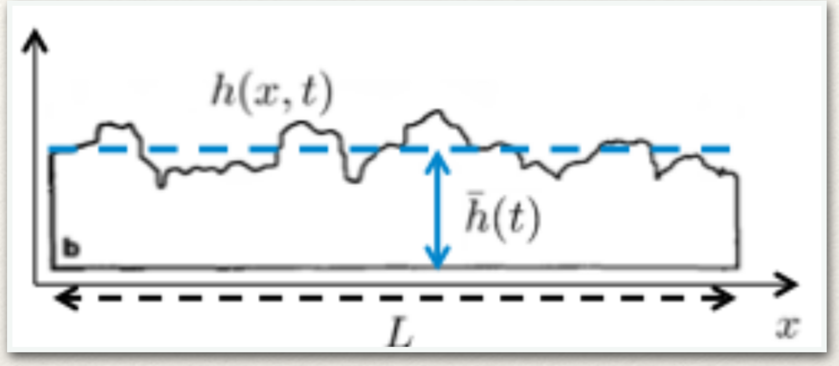

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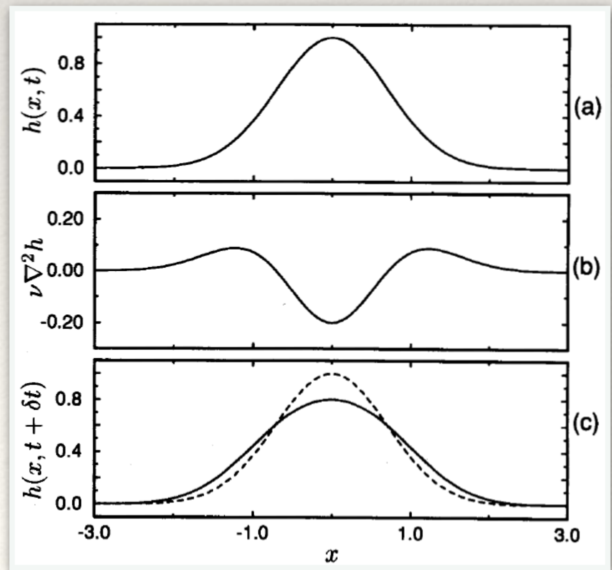

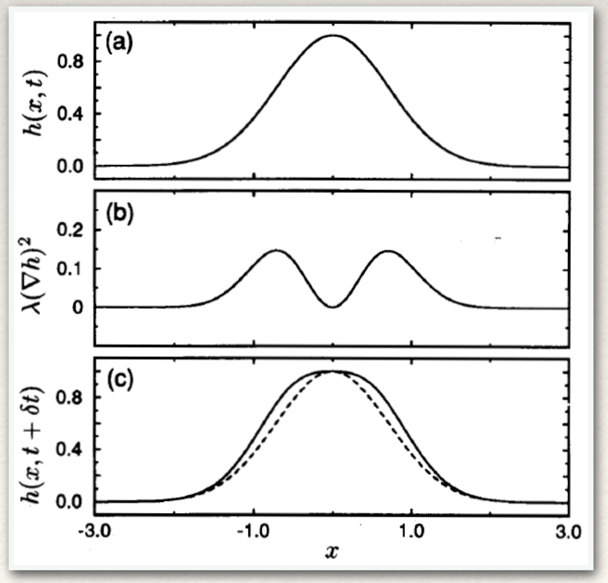

- The KPZ nonlinearity $(\lambda/2)(\nabla h)^2$, local surface normal growth, breaks the up-down symmetry $(h \to -h)$ of the EW equation.

A. L. Barabási and H. E. Stanley, *Fractal Concepts in Surface Growth*, CUP, 1995

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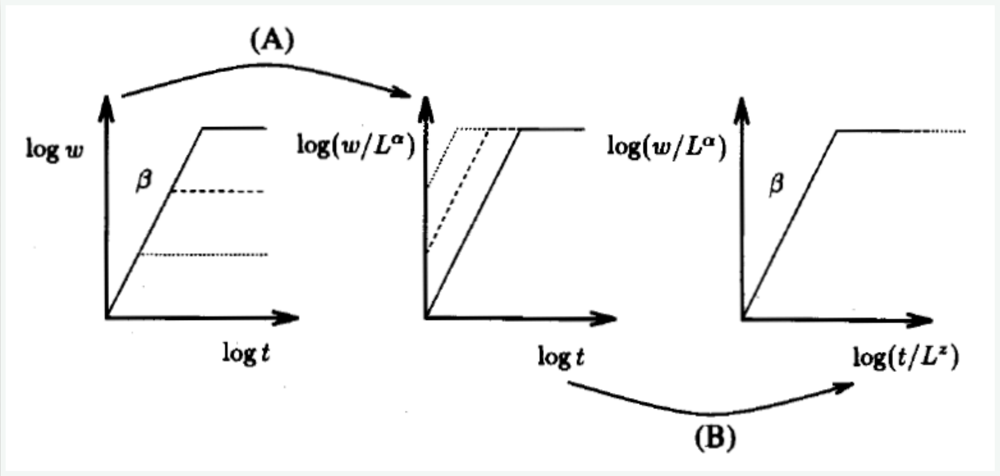

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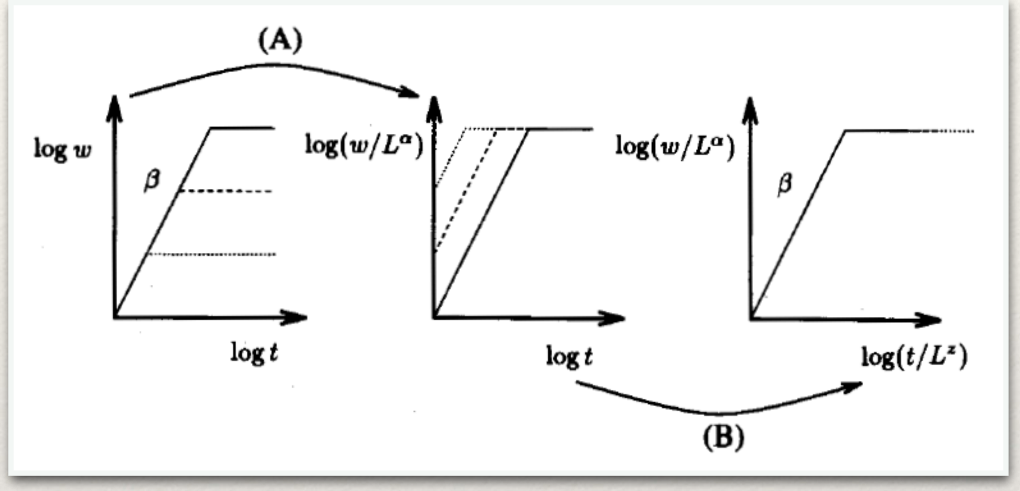

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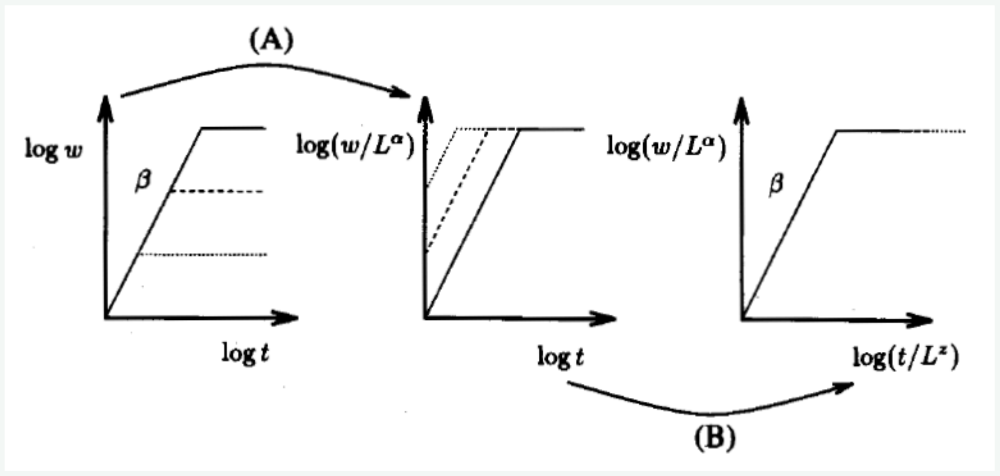

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- Apart from α and z, universality classes characterized by PDF of fluctuations around average growth t^{β}: Gaussian for EW class, Tracy-Widom (TW) for KPZ class.

- <u>Generic Scale Invariance (GSI)</u>: Similar to the critical dynamics of the Ising model, BUT it does not require setting parameters to specific (critical) values.

Continuum limit of a system of phase oscillators on a lattice

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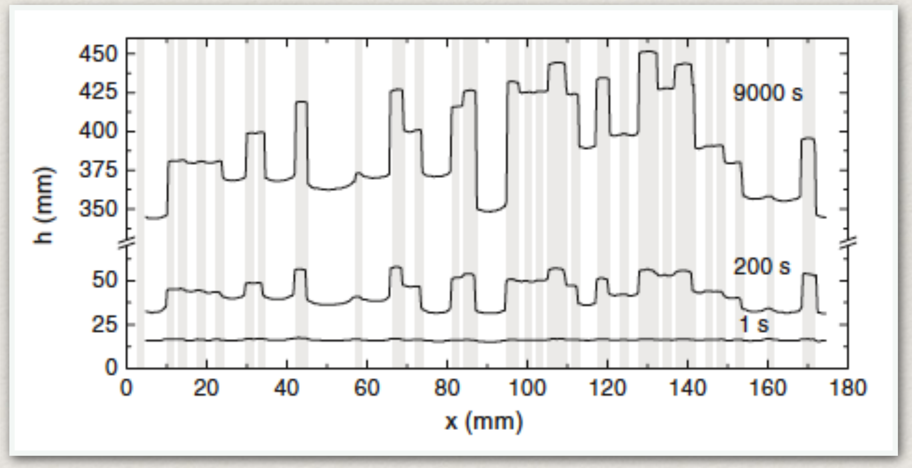

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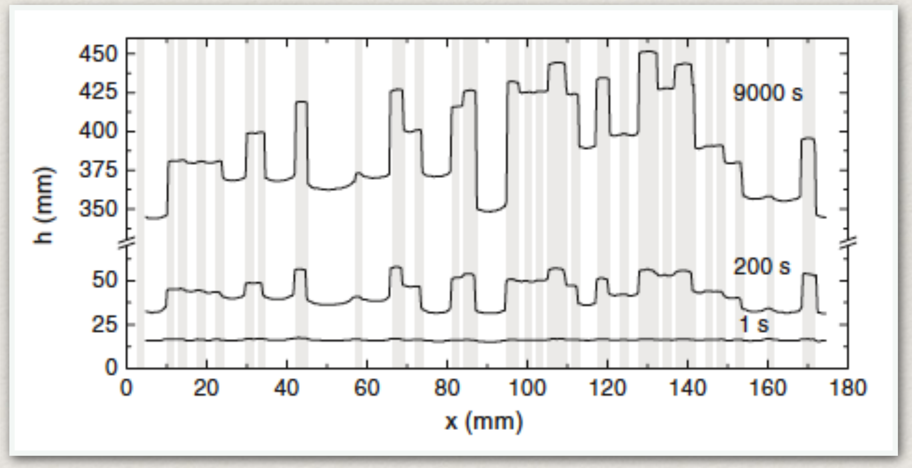

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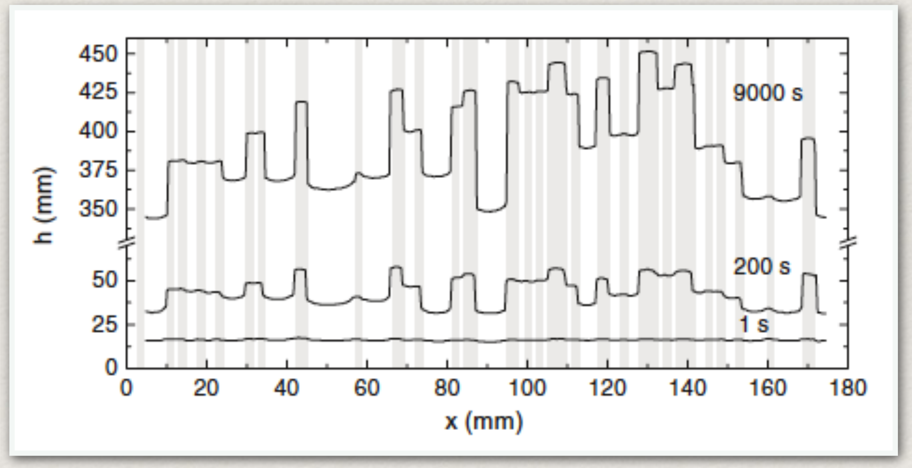

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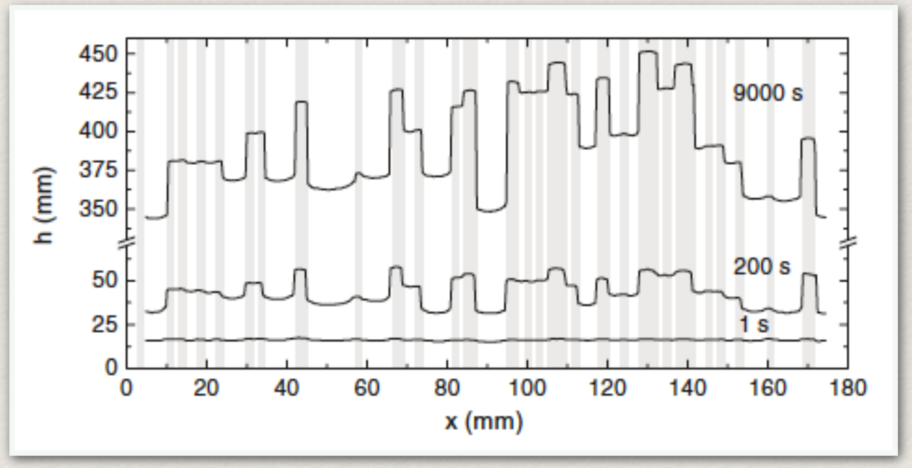

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- Generically, columnar KPZ. If $\Gamma^{(2)}(0) = 0$ ($\lambda = 0$): columnar EW (Larkin model).

Numerical results for a 1D Lattice of Kuramoto oscillators ($\delta = 0$)

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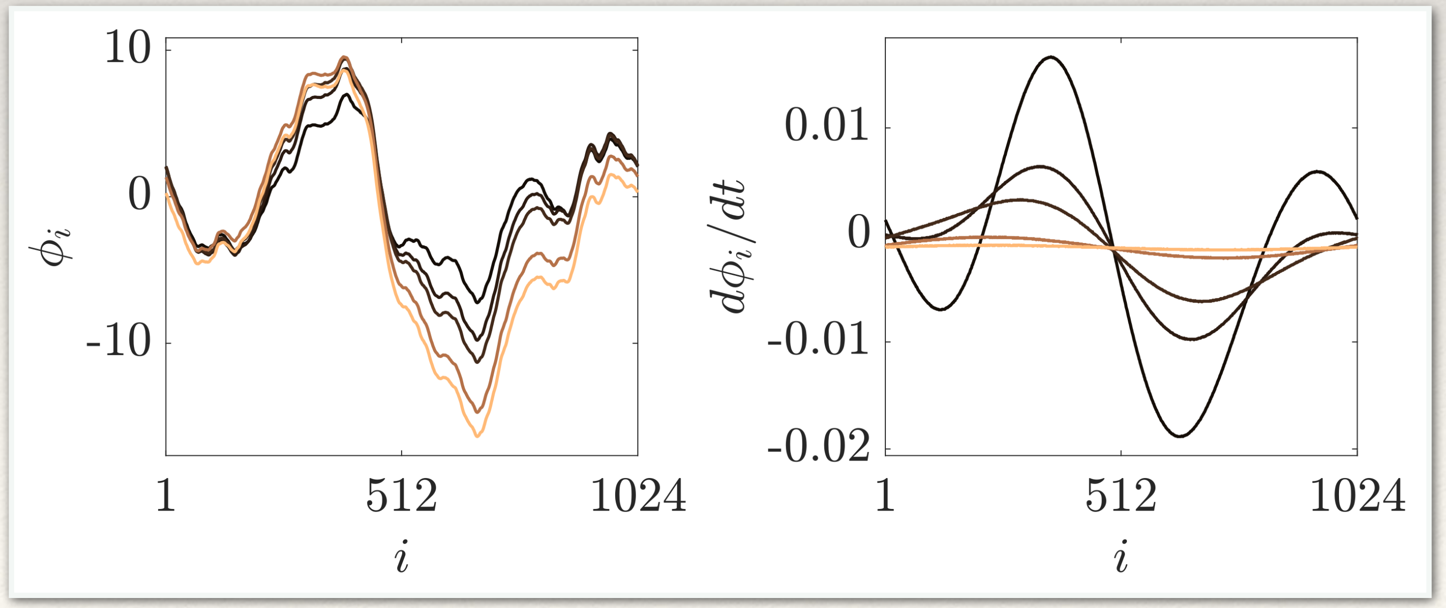

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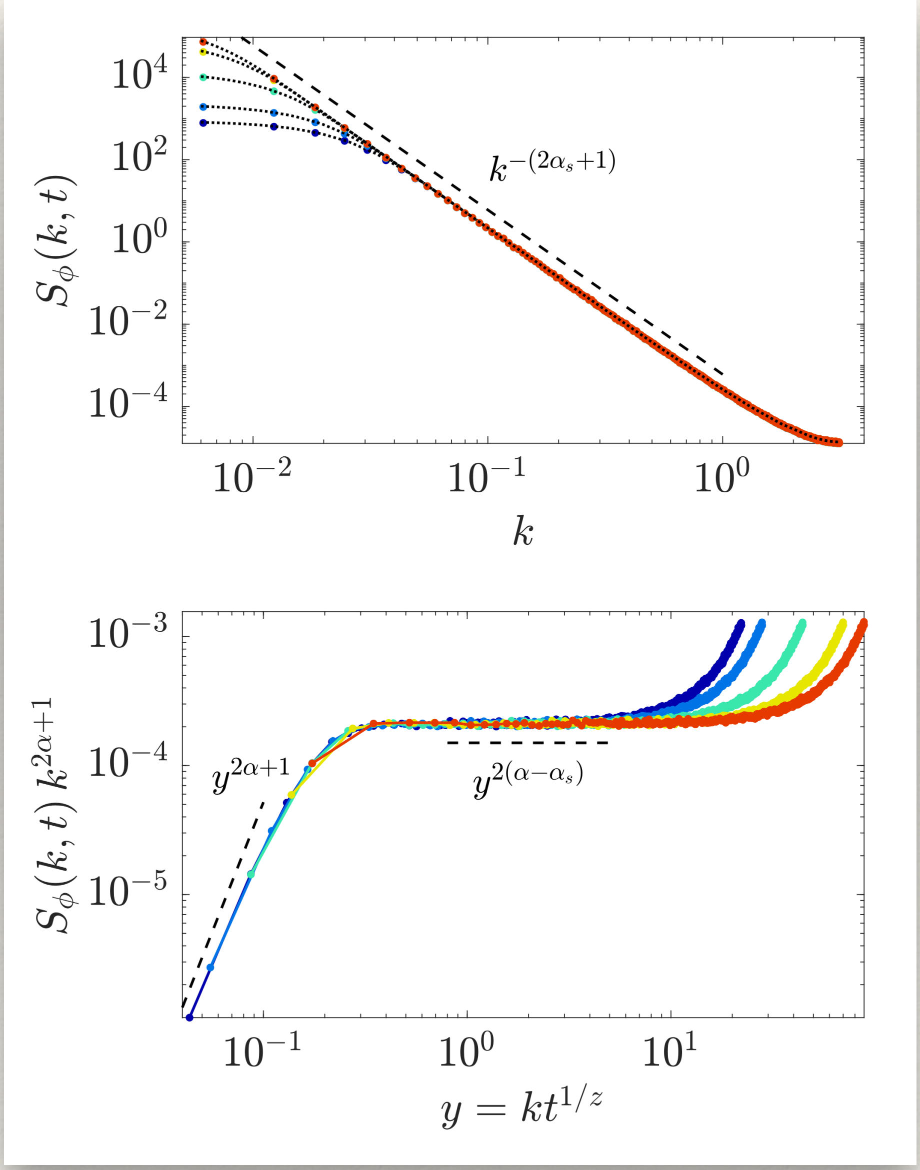

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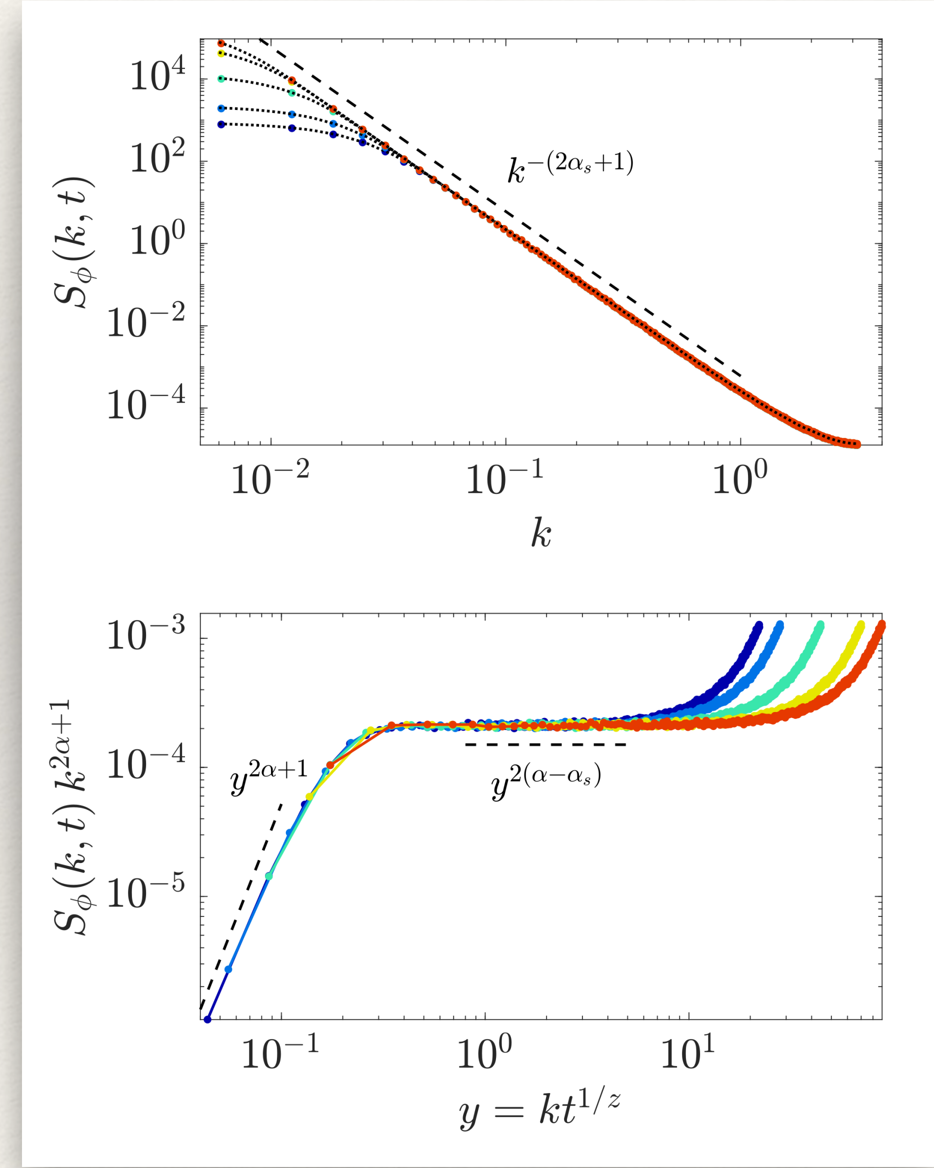

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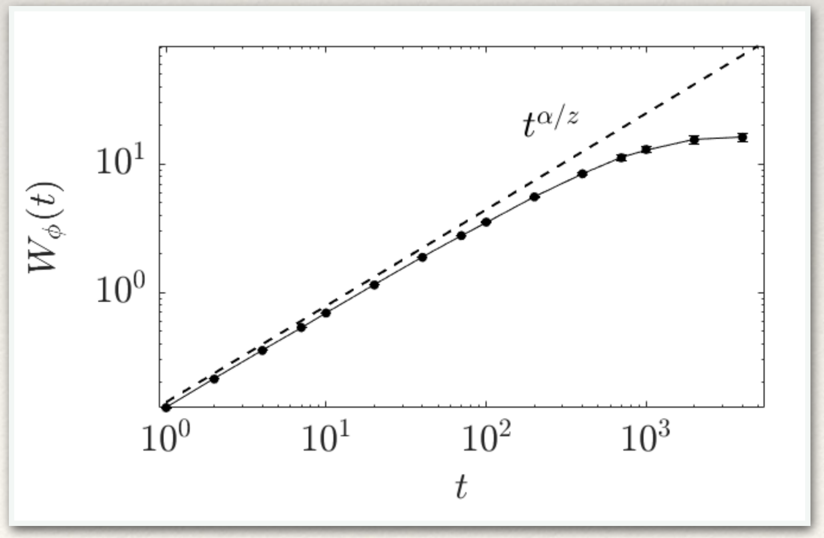

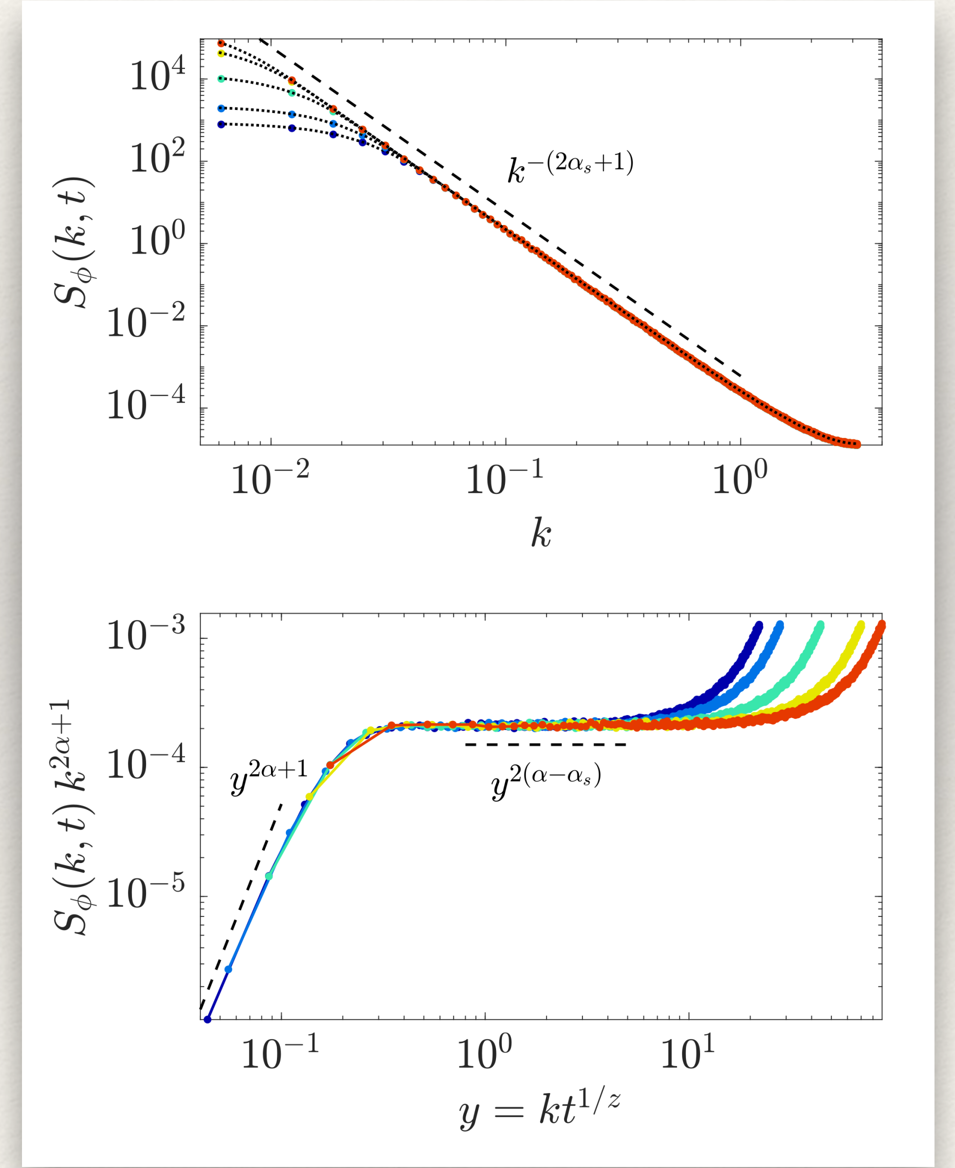

Numerical results for a 1D lattice of Kuramoto-Sakaguchi oscillators ($\delta \neq 0$)

General considerations and phenomenology

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Trajectories: phenomenological similarities

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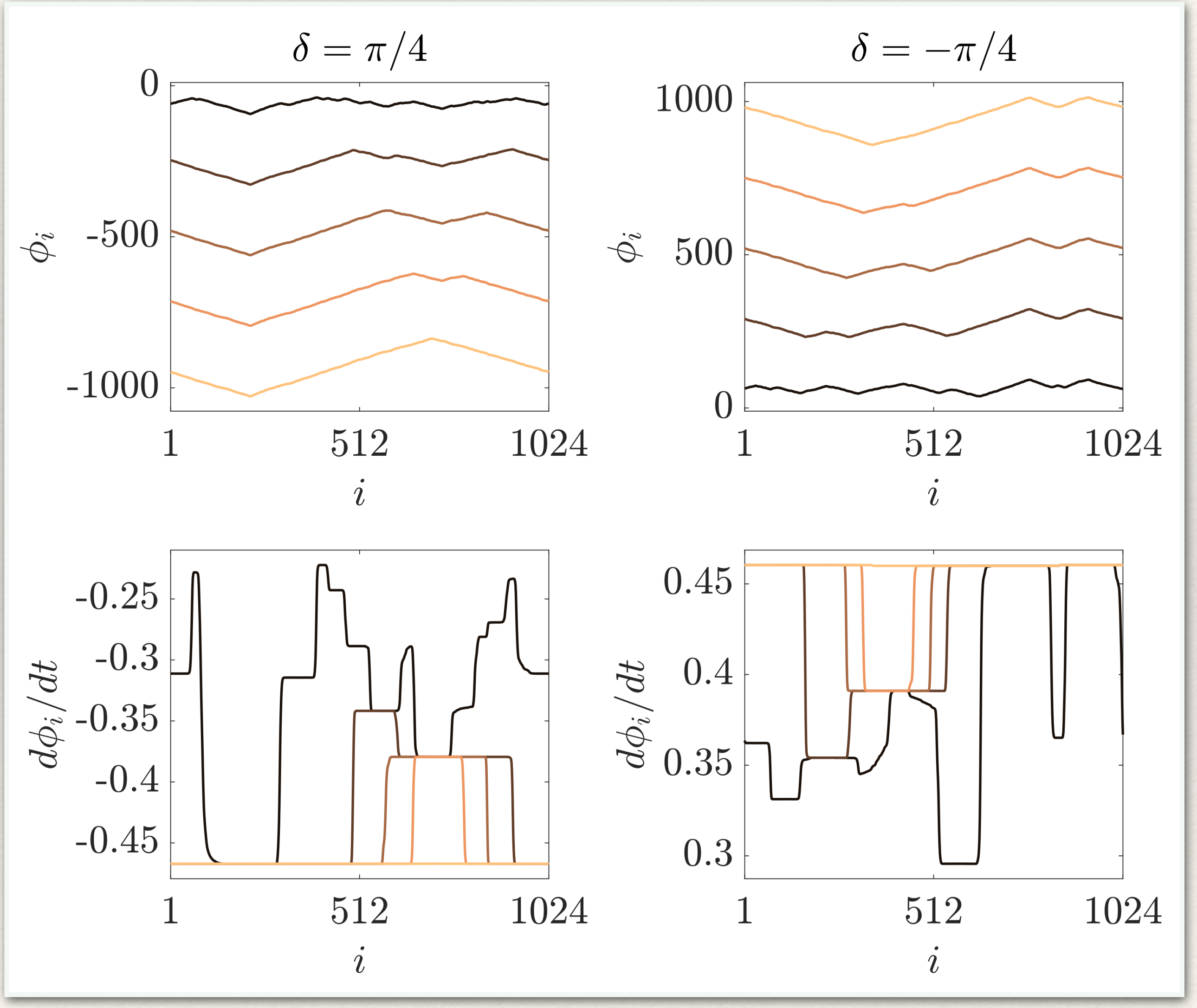

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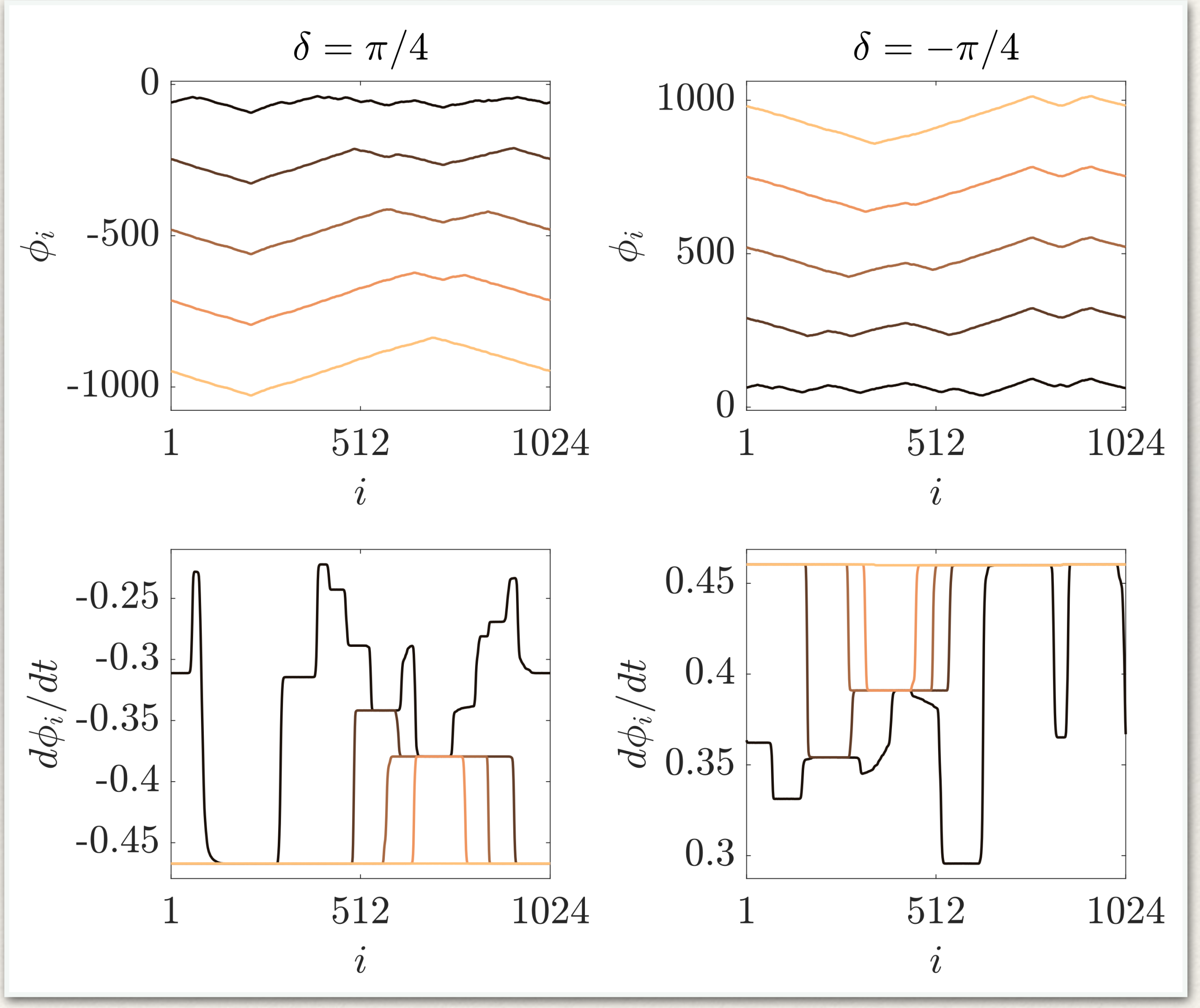
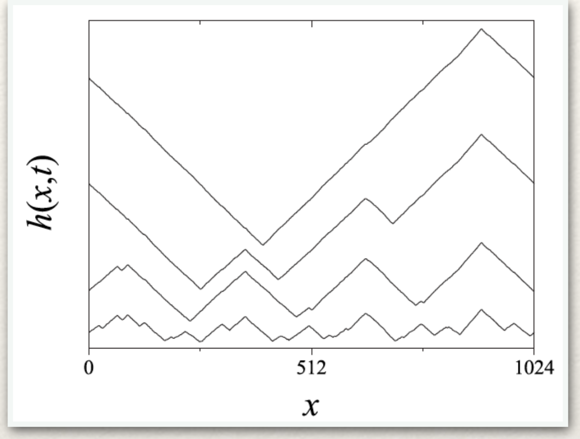


KPZ equation with columnar noise



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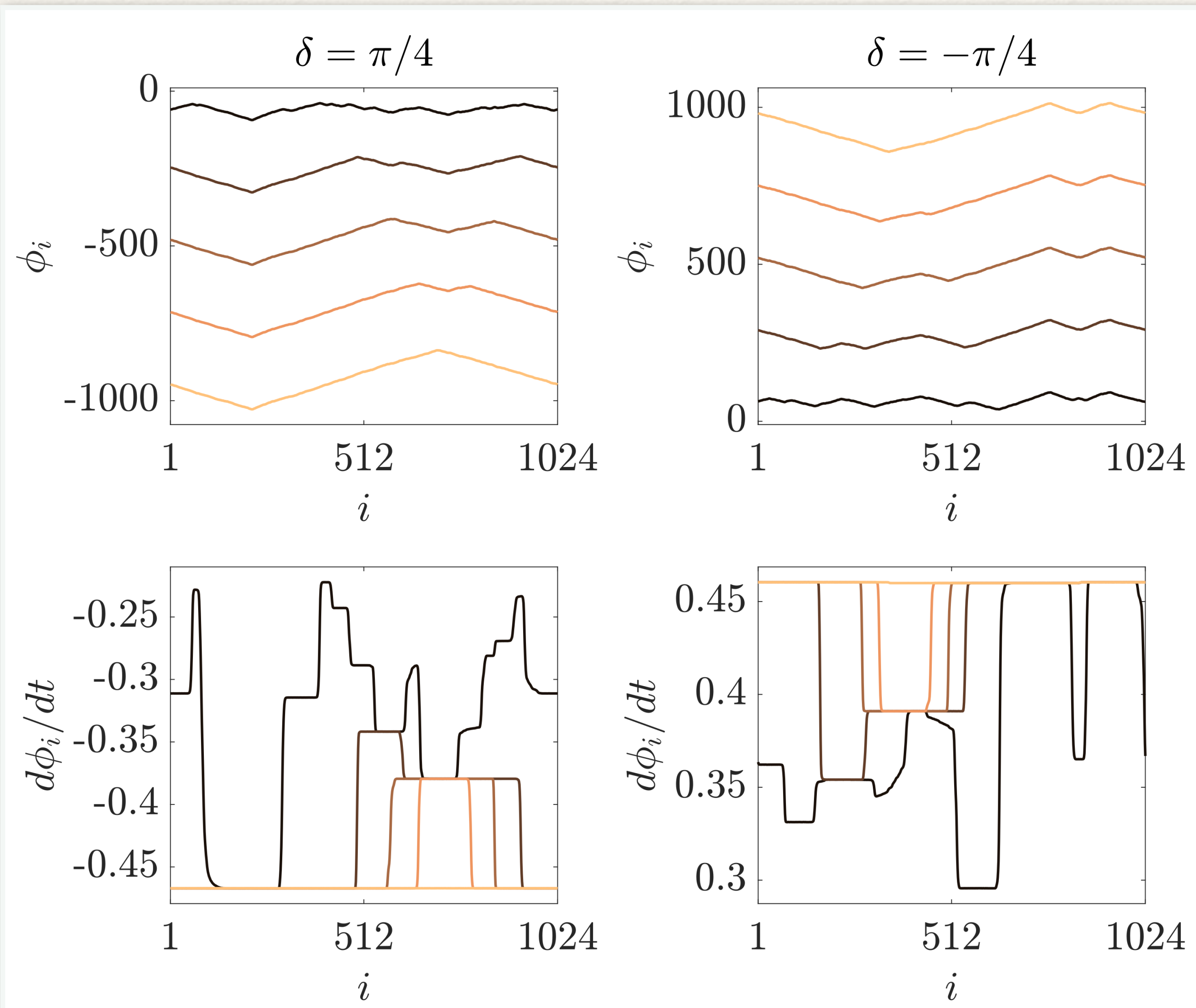

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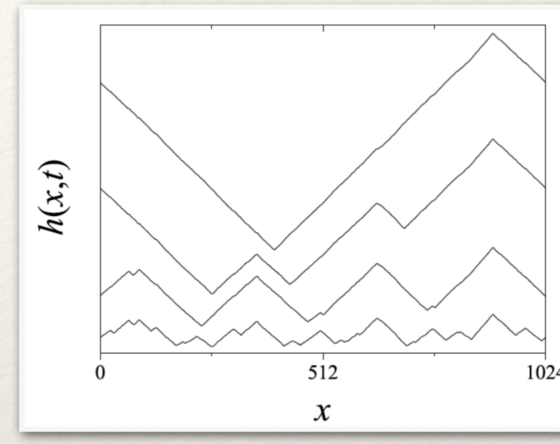

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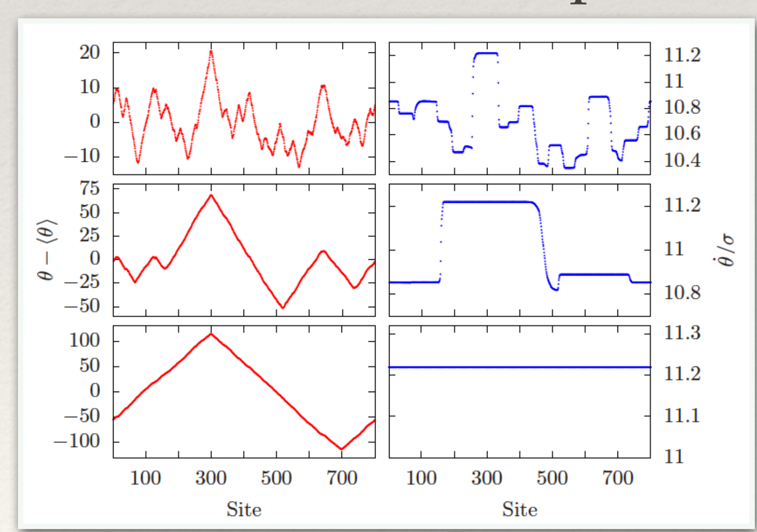

J. P. Moroney & P. R. Eastham,
Phys. Rev. Research 3, 043092 (2021)

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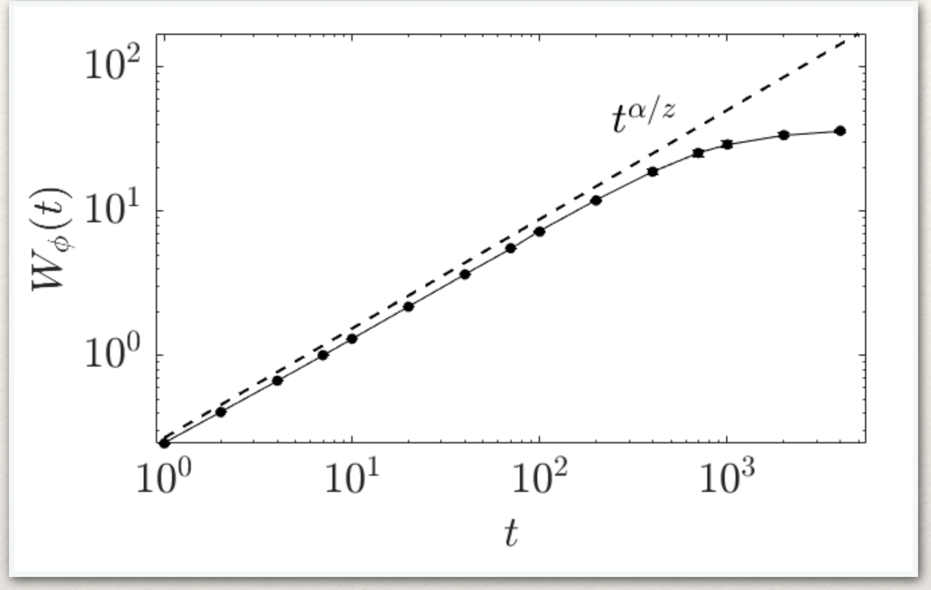

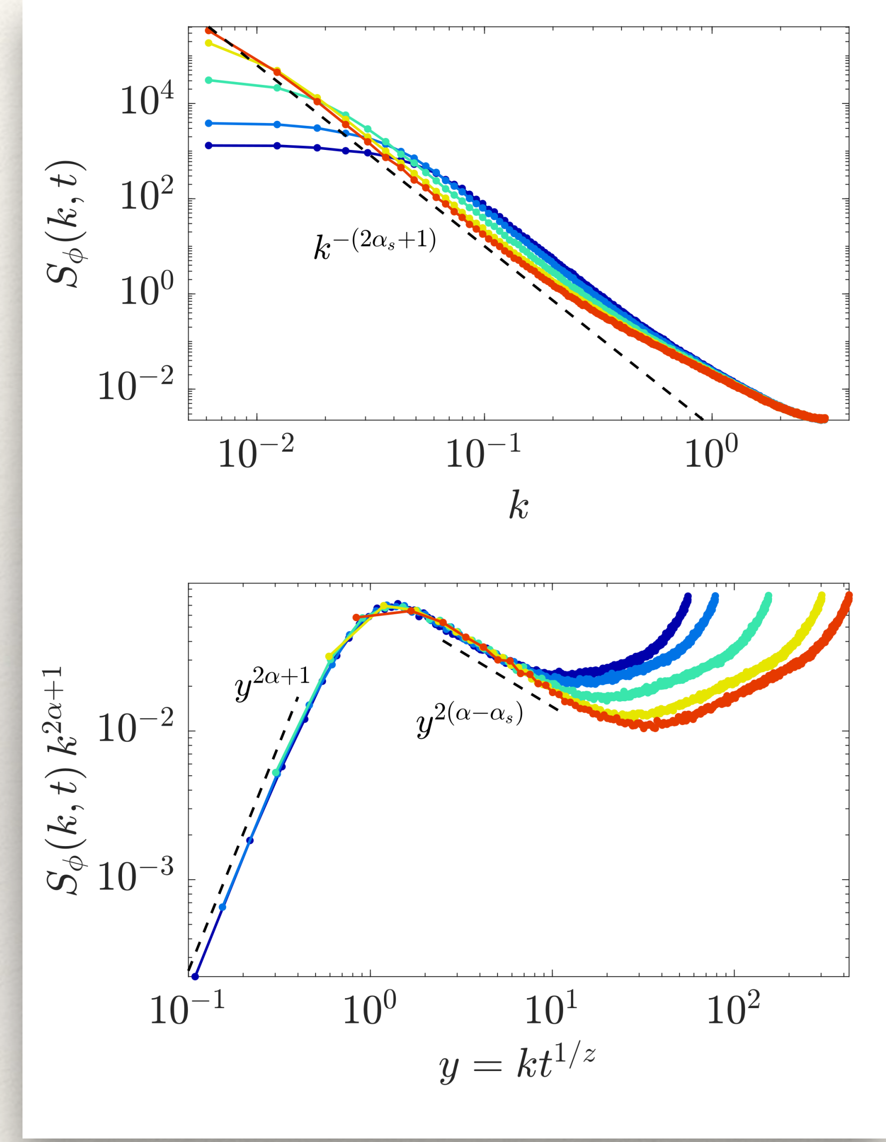

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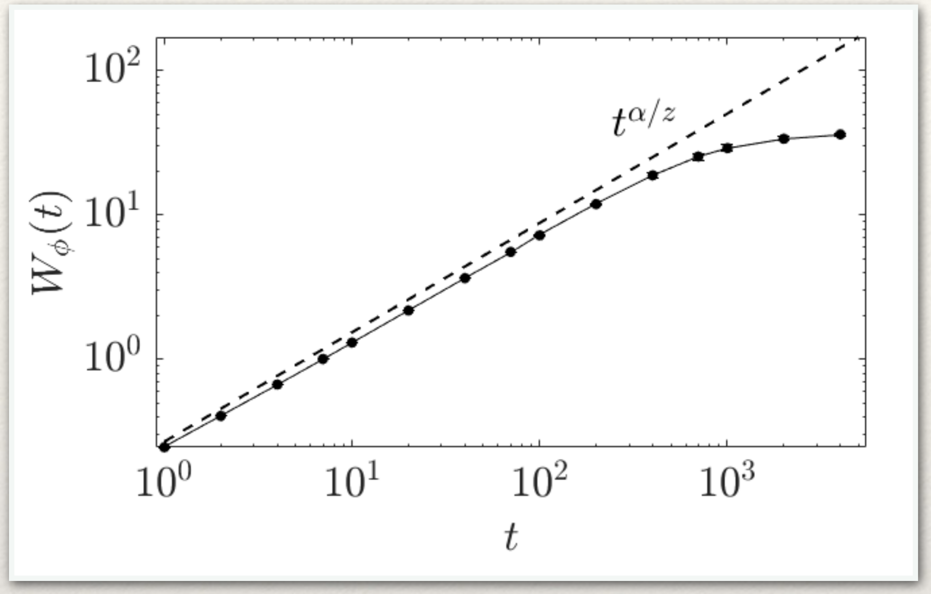

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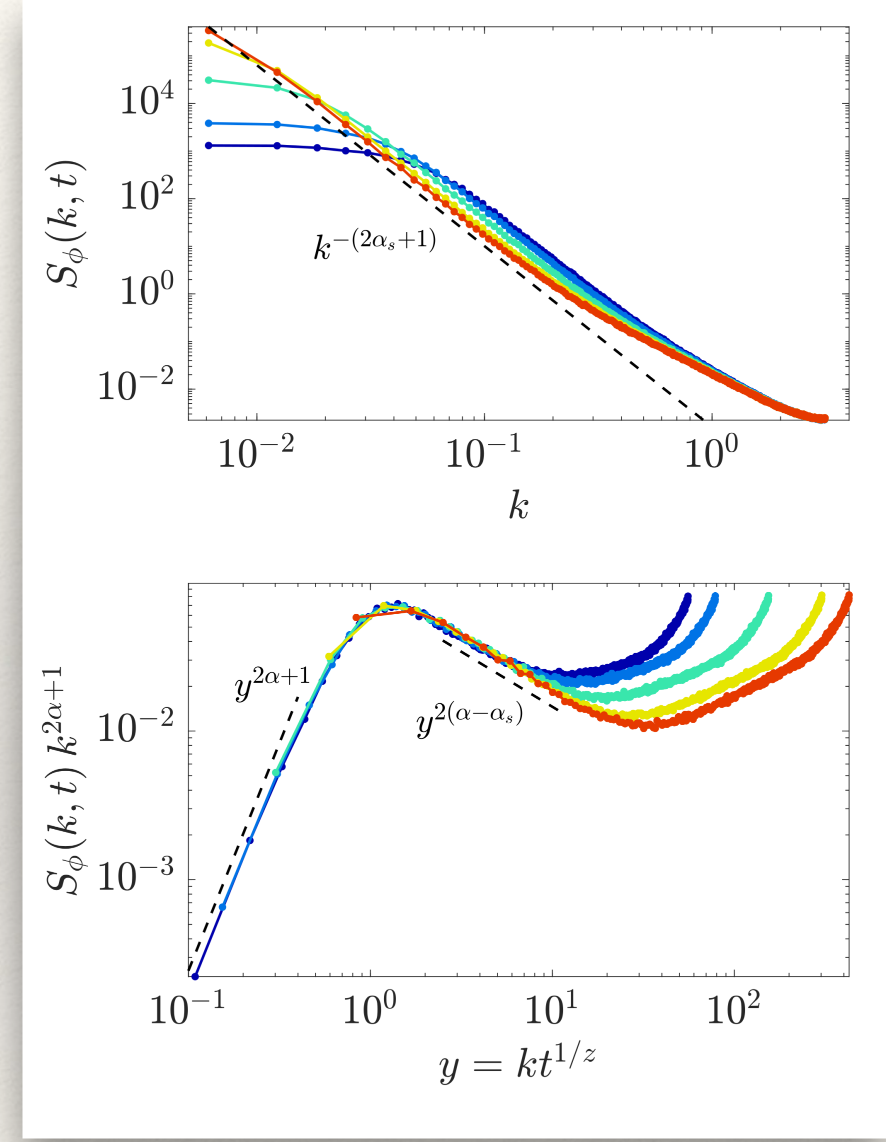

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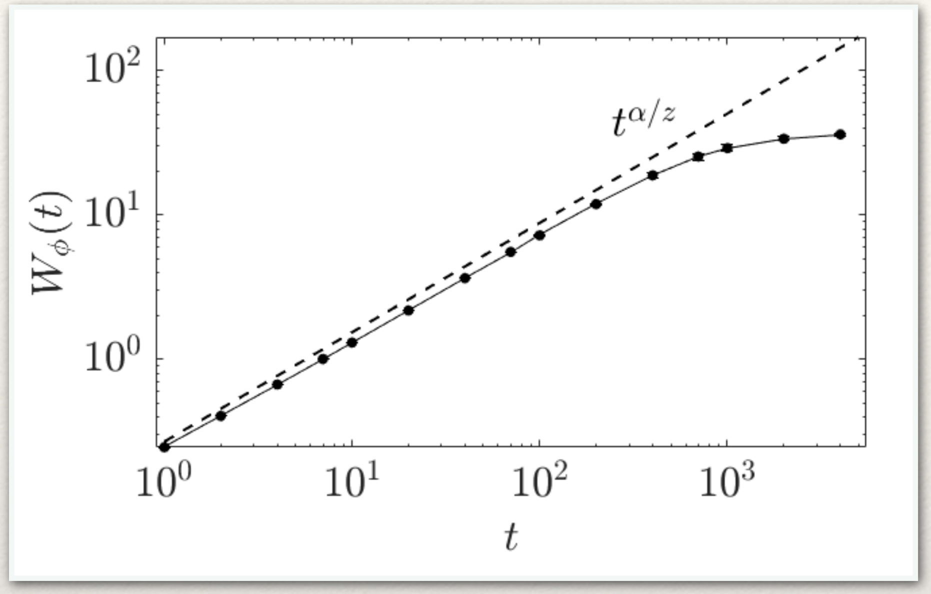

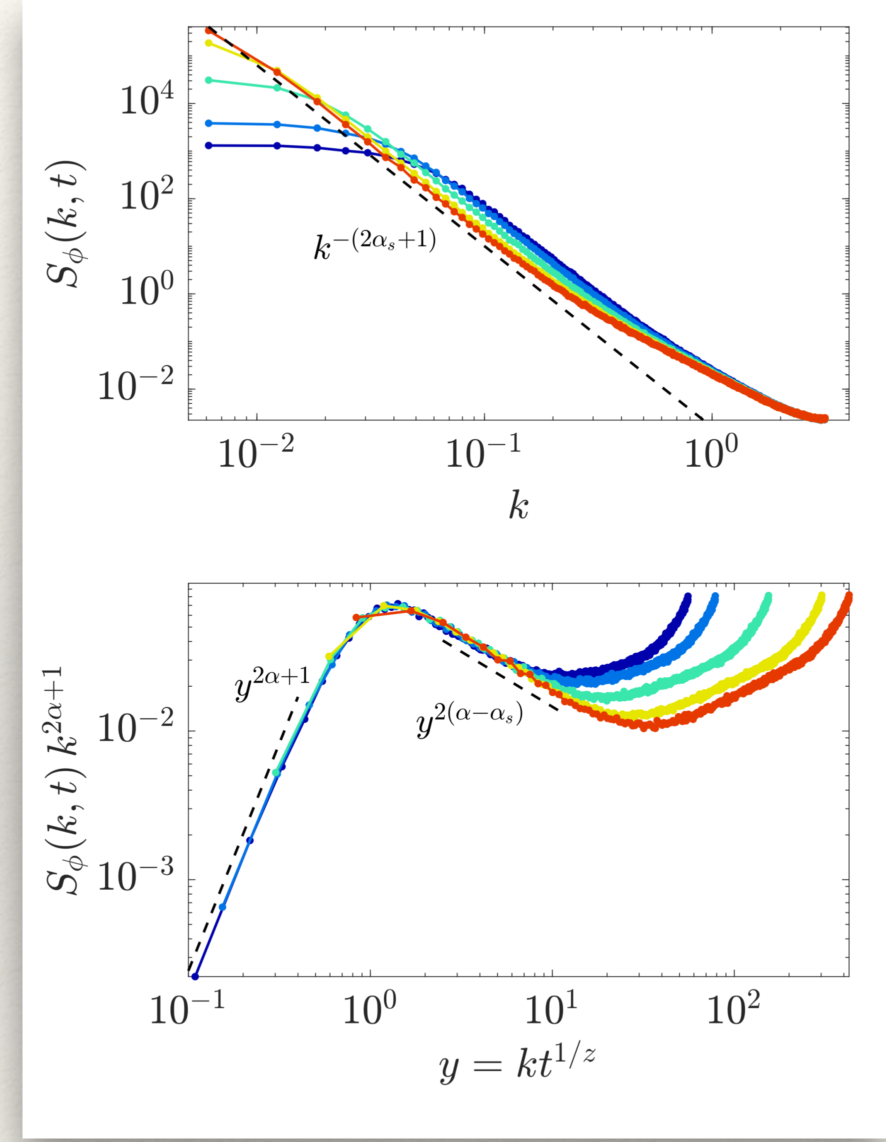

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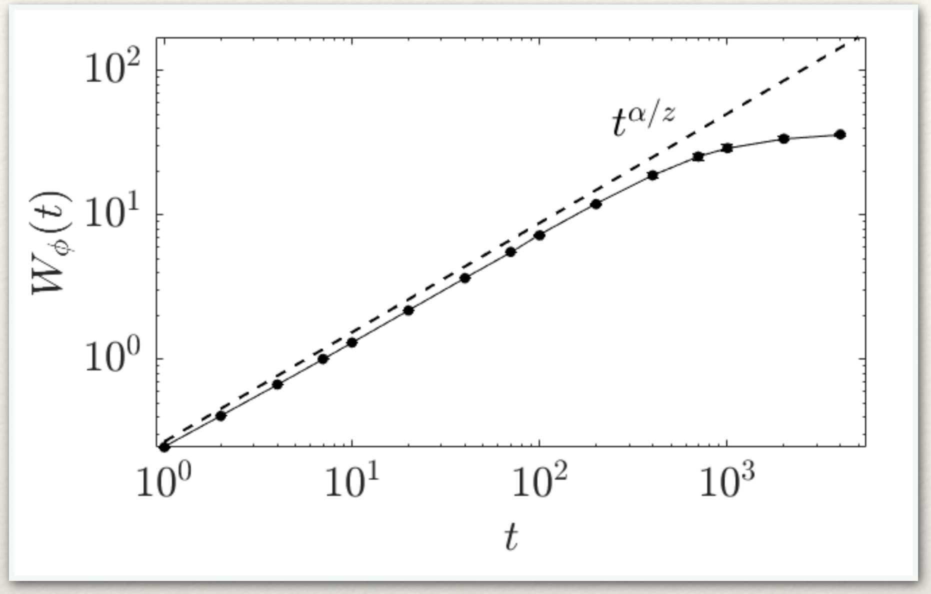

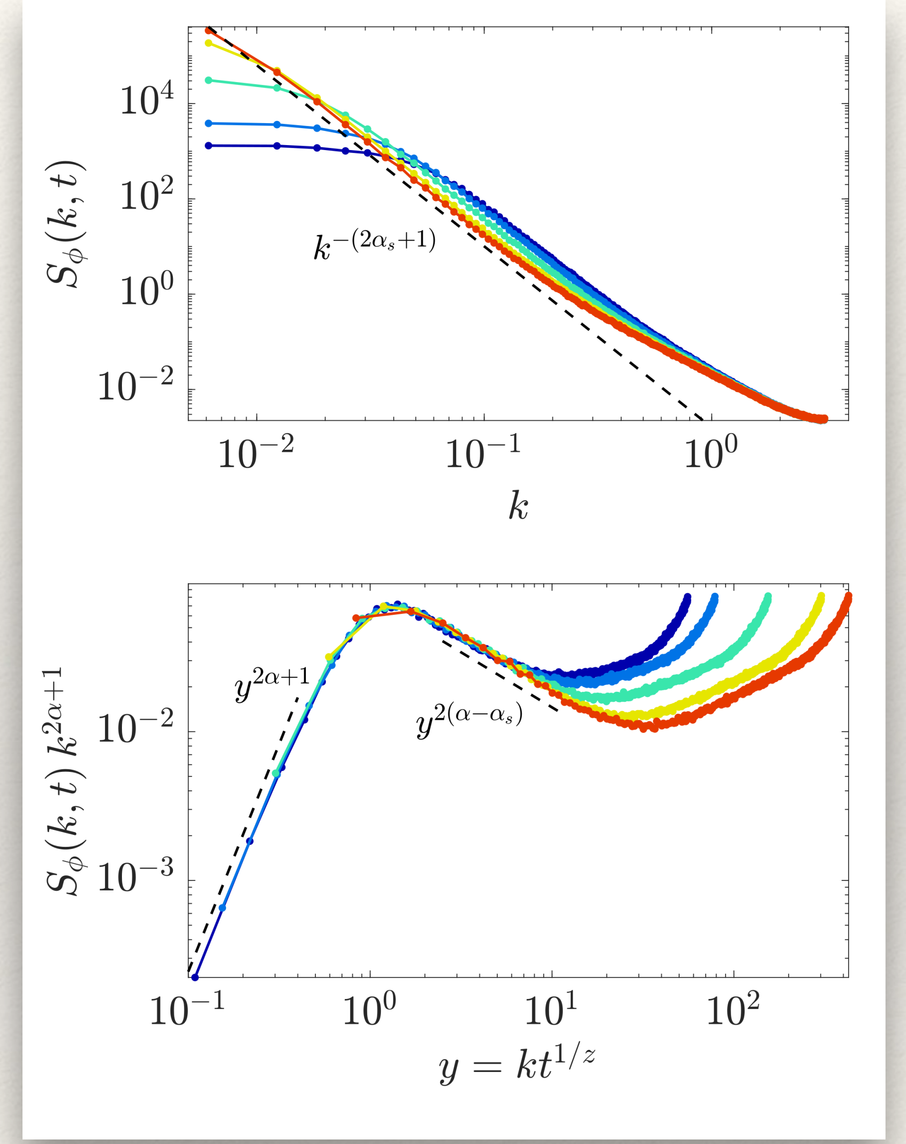

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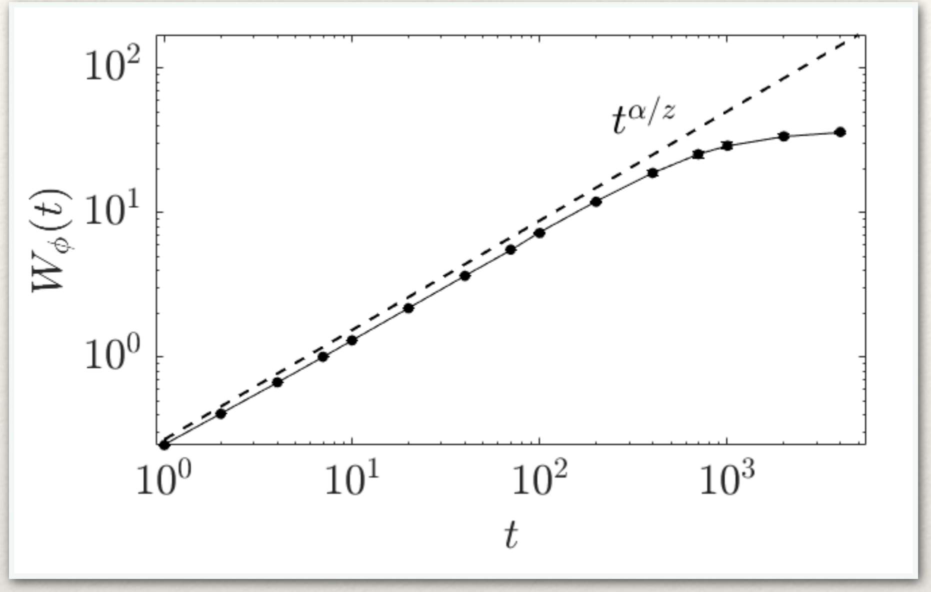

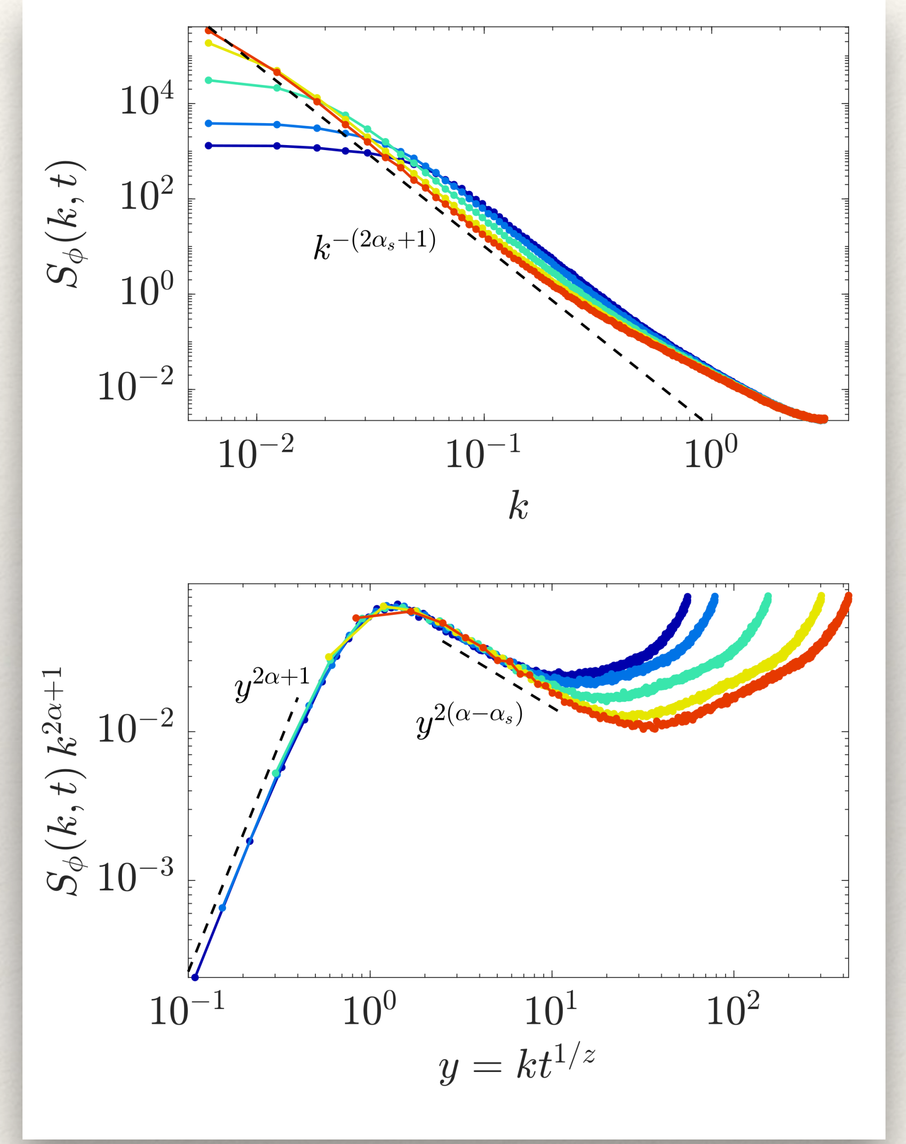

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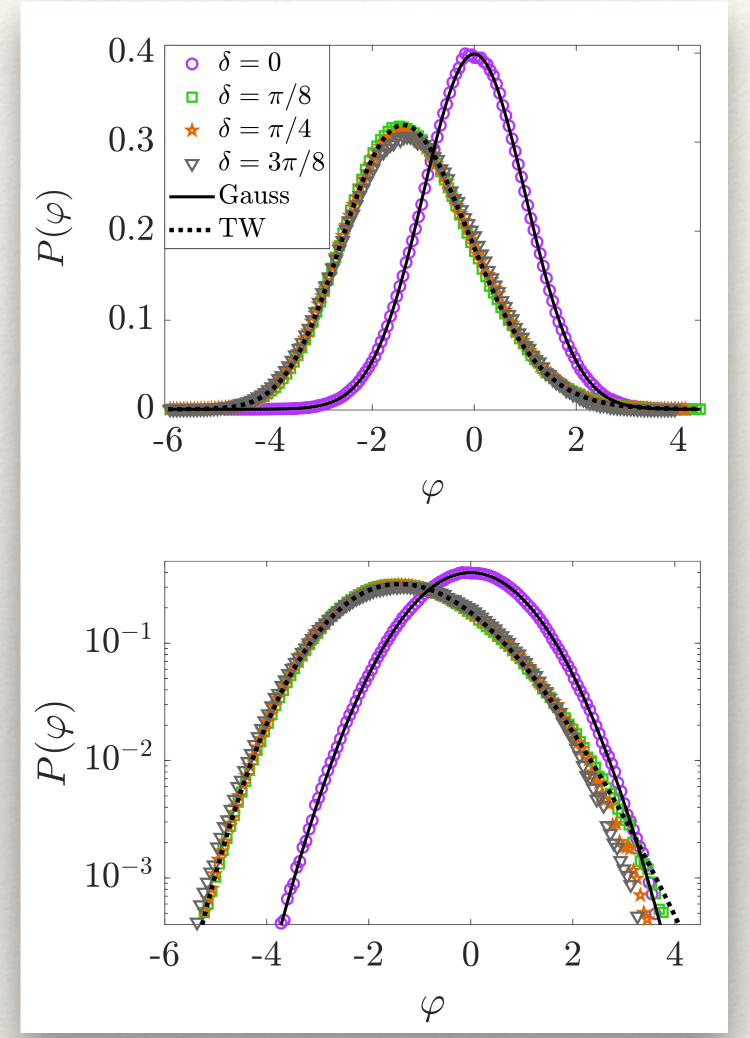

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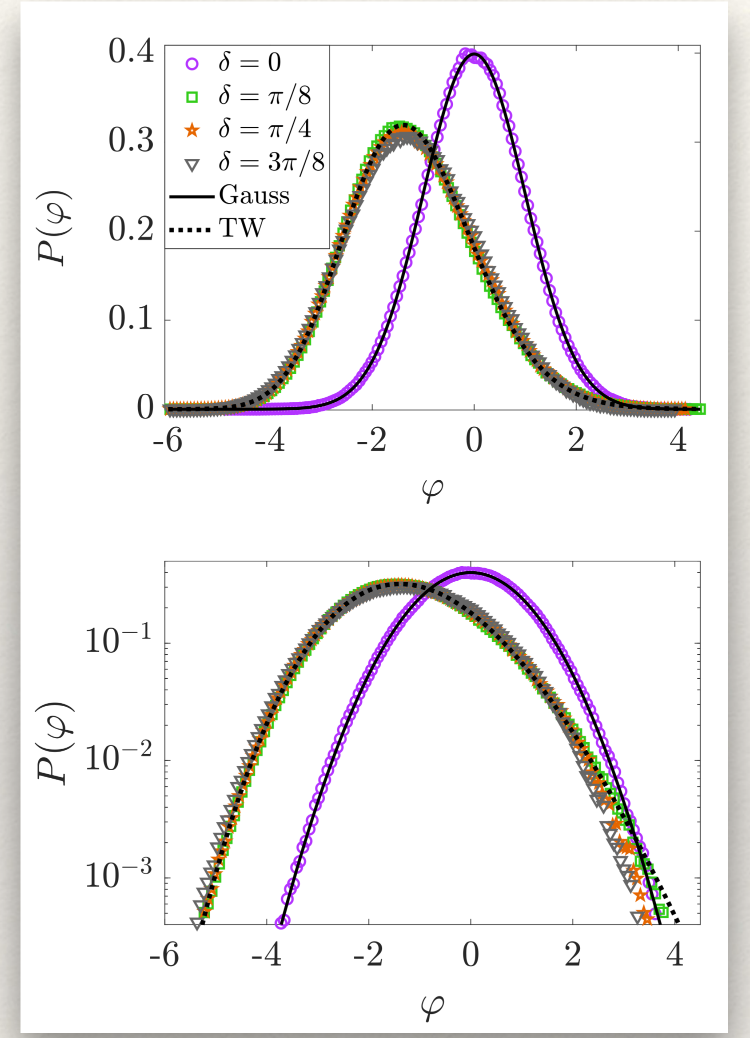

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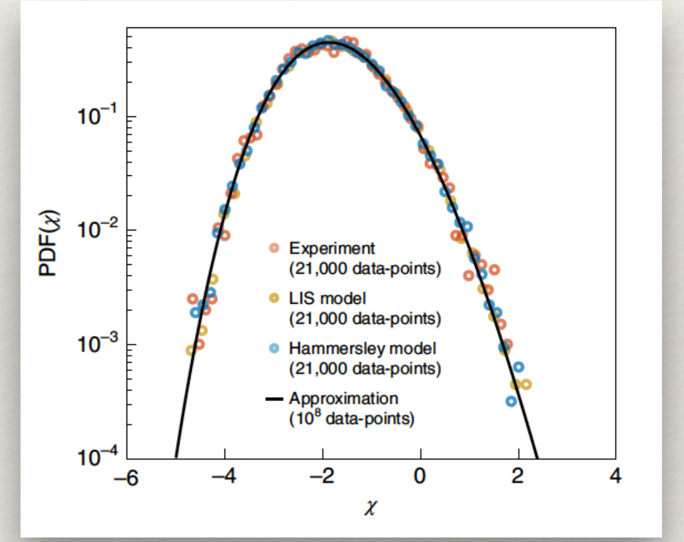

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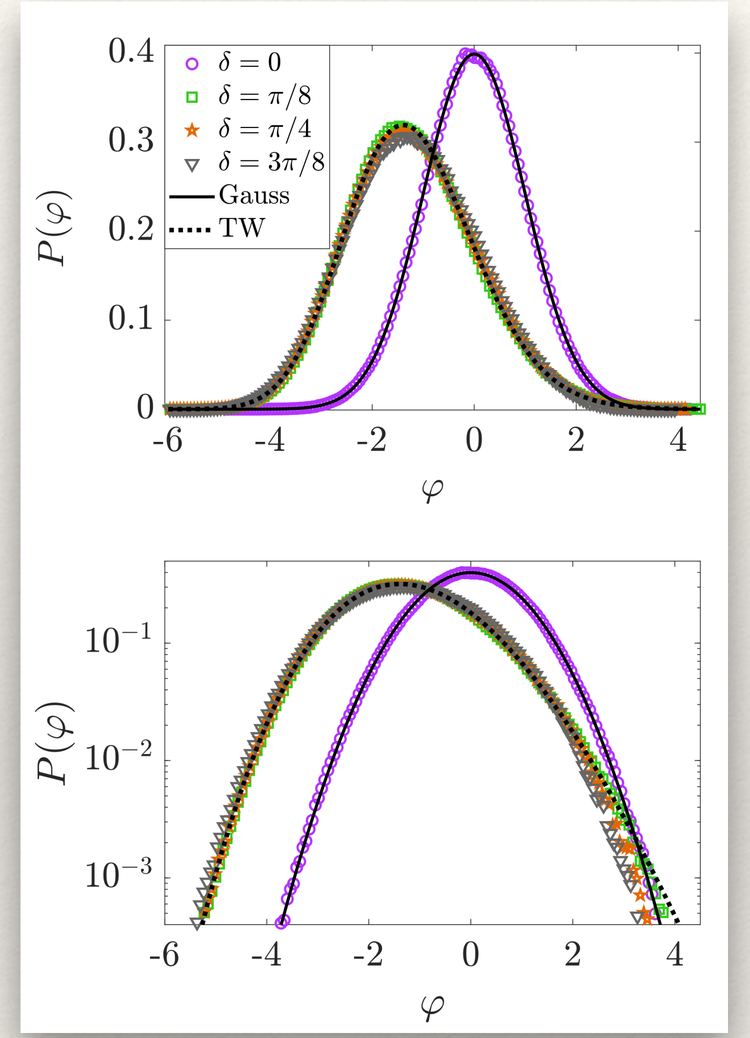

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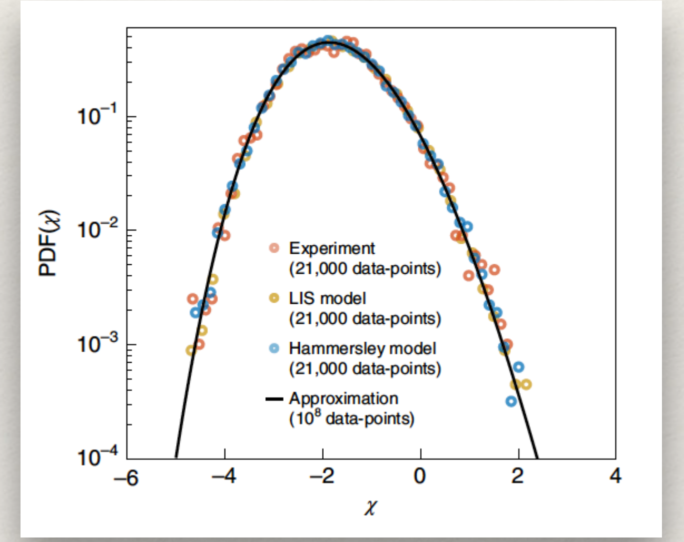

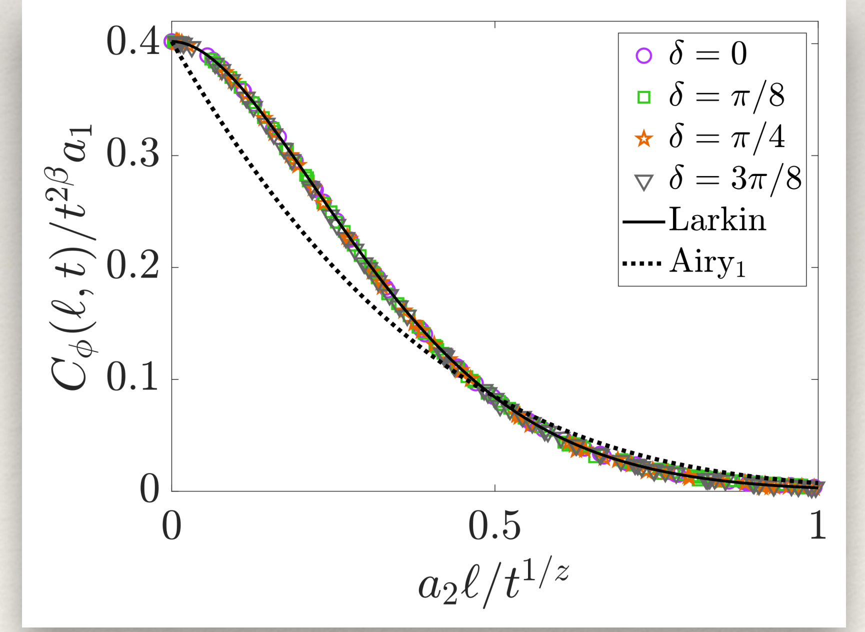

- Covariance $C_\phi(\mathbf{r}, t) \equiv \langle \overline{\phi(\mathbf{x}+\mathbf{r}, t)\phi(\mathbf{x}, t)} \rangle - \langle \bar{\phi}(t) \rangle^2$ (phase-phase correlations) of columnar EW ($\delta = 0$) obtained analytically, shown to be valid for all δ.

Conclusions and future work

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- In the latter cases the fluctuations are governed by TW statistics, paradigmatic in the critical behaviour of low-dimensional strongly correlated systems.

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- More details available (hopefully) soon on the arXiv…

Thank you!

Physical Review Research 5, 023047 (2023)

Funding: PID2021-123969NB-I00, PID2021-128970OAI00

MINISTERIO DE CIENCIA E INNOVACIÓN

AGENCIA ESTATAL DE INVESTIGACIÓN

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- For a slow spatial phase variation, as occurs for $K > K^*$ well into the synchronized regime, coarse-grained description:

$$\partial_t \phi(\mathbf{x}, t) = \omega^*(\mathbf{x}) + \nu \nabla^2 \phi(\mathbf{x}, t) + \frac{\lambda}{2} [\nabla \phi(\mathbf{x}, t)]^2$$