Entanglement entropy from non-equilibrium lattice simulations

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Quantum Entanglement in High Energy Physics

Based on: A. Bulgarelli and M.Panero [arXiv:2304.03311]

Motivation

- Entanglement can unveil universal properties of strongly coupled many-body systems (e.g. central charge).
- In gauge theories it can show whether the theory is in a confined or deconfined phase [Klebanov *et. al.*; 2007].
- Entanglement measures are notoriously difficult to calculate analytically, and numerical methods are limited by the high nonlocality of the observable.
- This motivates the search for efficient algorithms for the calculation of entanglement entropy and related quantities by means of Monte Carlo simulations.

Replica trick

- A common way to calculate Rényi entropies and other entanglement measurements is to exploit the replica trick [Calabrese, Cardy; 2004]

$$S_n = \frac{1}{1-n} \log \mathrm{Tr}\, \rho_A^n = \frac{1}{1-n} \log \frac{Z_n}{Z^n}$$

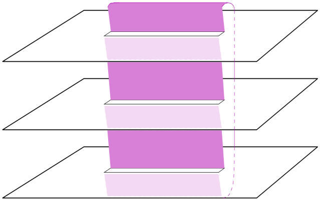


Image taken from [Cardy *et. al.*; 2007].

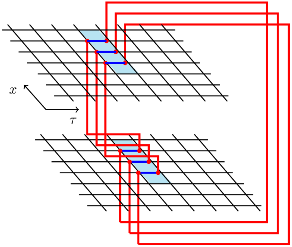
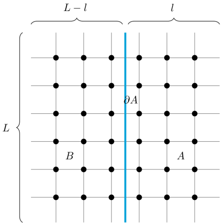


Image adapted from [Alba; 2016].

$$H = -\sum_{k=1}^{n} \left\{ \beta \sum_{\langle ij \rangle \not\perp \mathcal{C}} \sigma_i^{(k)} \sigma_j^{(k)} + \beta^{(k,k)} \sum_{\langle ij \rangle \perp \mathcal{C}} \sigma_i^{(k)} \sigma_j^{(k)} + \beta^{(k,k+1)} \sum_{\langle ij \rangle \perp \mathcal{C}} \sigma_i^{(k)} \sigma_j^{(k+1)} \right\}$$

Entropic c-function

- Problem: Rényi entropies are always UV divergent.
- We will consider a slab bipartition of the lattice, so that the entangling surface ∂A does not depend on l.



- In this setup the entropic c-function is UV finite and encodes all the universal information contained in the Rényi entropies

$$C_n = \frac{l^{D-1}}{|\partial A|}\frac{\partial S_n}{\partial l}$$

- We work at zero temperature ($L_\tau \gg L$), where $S_n(l) = S_n(L-l)$; therefore we use a symmetrized version of the entropic c-function

$$C_n = \frac{\left[\frac{L}{\pi}\sin\left(\frac{\pi l}{L}\right)\right]^{D-1}}{|\partial A|}\frac{\partial S_n}{\partial l}$$

Ratios of partition functions

- Also $\partial S_n / \partial l$ can be written in terms of a ratio of partition functions. Using a lattice regularization

$$\frac{\partial S_n}{\partial l} \simeq \frac{1}{1-n} \frac{1}{a} \log \frac{Z_n(l+a)}{Z_n(l)}$$

- Typical lattice approaches used for calculations of the entropic c-function (see *e.g.* [Buividovich, Polikarpov; 2008]) suffer from a bad signal-to-noise ratio [Jokela *et. al.*; 2023].

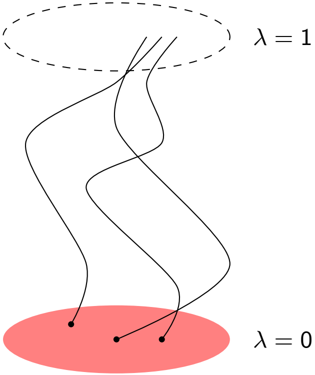
- In recent years the Turin group has exploited Jarzynski's equality [Jarzynski; 1996] to perform high-precision lattice calculations [Caselle *et. al.*; 2016][Caselle *et. al.*; 2018][Francesconi *et. al.*; 2020][Caselle *et. al.*; 2022].

- It is becoming increasingly clear that non-equilibrium Monte Carlo calculations based on this theorem can be a reliable tool for studies of the entanglement entropy, too [Alba; 2016][D'Emidio; 2019][Zhao *et. al*; 2021][Da Liao *et. al.*; 2023].

Jarzinski's theorem

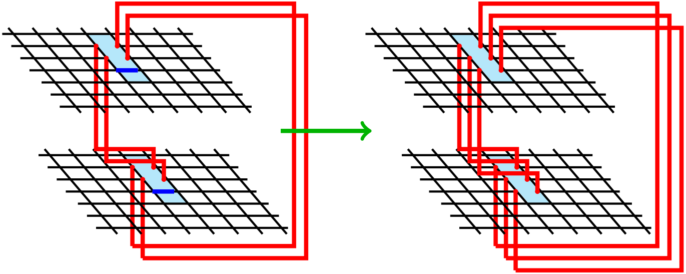
- Jarzynski's theorem [Jarzynski; 1996] is an exact result that connects averages of out-of-equilibrium trajectories of a statistical system to equilibrium free energies.
- The theorem is valid both for real and Monte Carlo time evolution.
- Consider the one parameter evolution $H_{\lambda=0} \to H_{\lambda=1}$. Jarzynki's theorem states that

$$\left\langle \exp\left(-\int \beta\delta W\right)\right\rangle = \frac{Z_{\lambda=1}}{Z_{\lambda=0}}$$



Our algorithm

$$\frac{\partial S_n}{\partial l} \simeq \frac{1}{1-n}\frac{1}{a}\log\frac{Z_n(l+a)}{Z_n(l)}$$



$$H = -\sum_{k=1}^{n}\left\{\beta \sum_{\langle ij\rangle \not\perp \mathcal{C}} \sigma_i^{(k)}\sigma_j^{(k)} + \beta^{(k,k)} \sum_{\langle ij\rangle \perp \mathcal{C}} \sigma_i^{(k)}\sigma_j^{(k)} + \beta^{(k,k+1)} \sum_{\langle ij\rangle \perp \mathcal{C}} \sigma_i^{(k)}\sigma_j^{(k+1)}\right\}$$

Theoretical results for $2D$ CFTs

- The theoretical prediction for a CFT on a cylinder of spatial length L is ($c = \frac{1}{2}$ for the Ising model)

$$C_2(x) = \frac{c}{8}\cos(\pi x) \qquad\qquad x = \frac{l}{L}$$

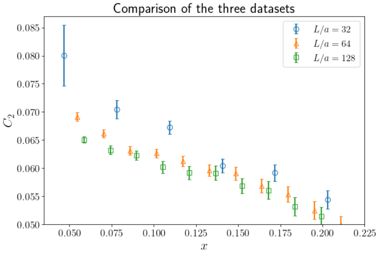
- This result holds in the scaling limit $L, l \gg 1$, while for finite sizes scaling corrections can be relevant.

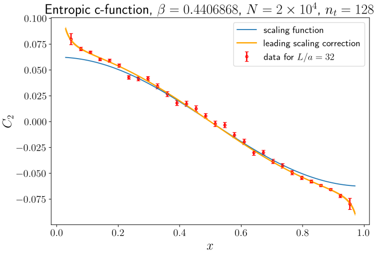
- The general theory of unusual corrections to scaling of the entanglement entropy was developed in [Calabrese, Cardy; 2010].

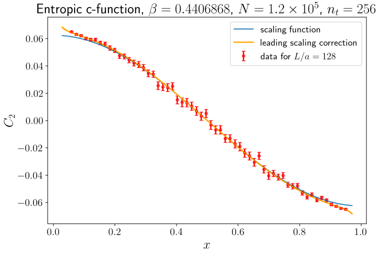
- In the case of the $D = 2$ Ising model one expects

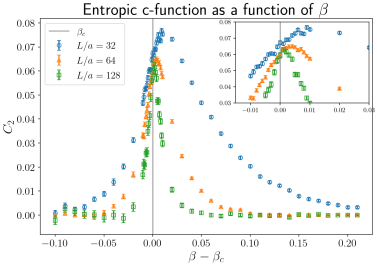
$$C_2(x) = C_2^{\mathrm{CFT}}(x) + \frac{k}{2L}\cot(\pi x)$$

Benchmark: Ising $2D$









Some models in $D = 3$

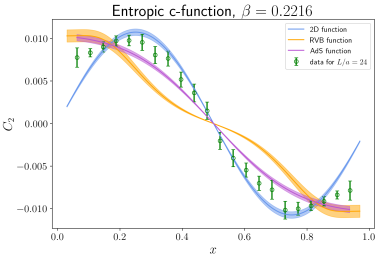
- For the $D = 3$ Ising model no analytical solution is known and only few numerical studies are present in literature [Inglis, Melko; 2013] [Kulchytskyy *et. al.*; 2019].
- We compared our numerical results at the critical point with three different models:
  - the $2D$ function
  - a function proposed in a study of resonance-valence-bond (RVB) dimers [Stéphan *et. al.*; 2012]
  - a function derived in [Chen *et. al.*; 2015] in a holographic setup using the Ryu-Takayanagi formula [Ryu, Takayanagi; 2006]

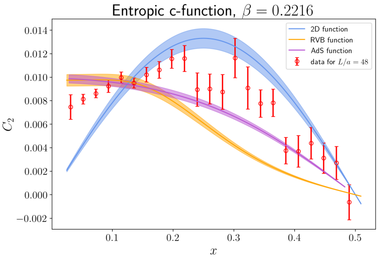
$$S_{2;2D}(x;c,k) = c\log(\sin(\pi x)) + k$$

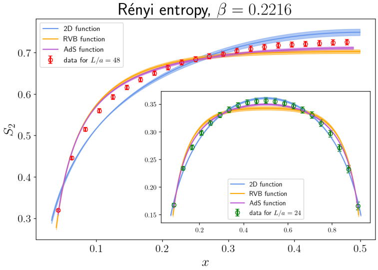
$$S_{2;RVB}(x;c,k) = -2c\log\left\{\frac{\eta(\tau)^2}{\theta_3(2\tau)\theta_3(\tau/2)}\frac{\theta_3(2x\tau)\theta_3(2(1-x)\tau)}{\eta(2x\tau)\eta(2(1-x)\tau)}\right\} + k$$

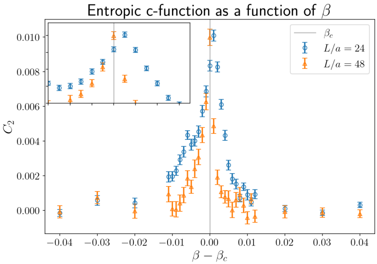
$$S_{2;AdS}(x;c,k) = c\chi(x)^{-\frac{1}{3}}\left\{\int_0^1 \frac{\mathrm{d}y}{y^2}\left(\frac{1}{\sqrt{P(\chi(x),y)}} - 1\right) - 1\right\} + k$$

Results for Ising 3D









Conclusions and future prospects

- Our data for the $2D$ Ising model are in perfect agreement with the CFT prediction.
- In $3D$ we showed that our data are well described by a function extracted from a holographic model and that in a slab geometry the entropic c-function is monotonically decreasing.
- In both cases we obtained precise results in a small amount of time (<800 CPU hours for the largest lattice size both in $D = 2, 3$).
- This algorithm can be generalized to arbitrary spin models and gauge theories.

Future work:

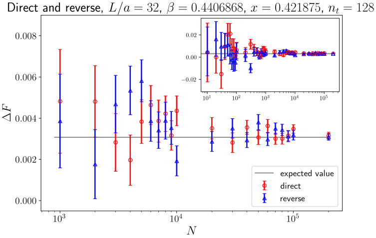
- Exploit the duality properties of the $3D$ Ising model to study the entanglement content of the $\mathbb{Z}_2$ gauge theory.
- Extension to non-Abelian gauge theories?

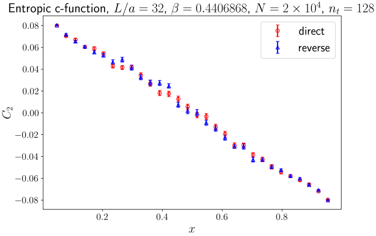
Appendix

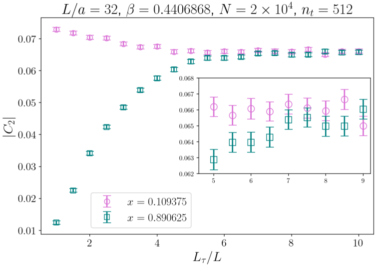
Some numerical details

- For our simulations we adapted the code found in [Komura, Okabe; 2014], implementing the replica space and Jarzynski's algorithm.
- The code is written in CUDA C to achieve high parallelization.
- We obtained precise results in a small amount of time: data for $L = 128$ required approximatively 750 hours on on the CINECA Marconi100 accelerated cluster, based on IBM Power9 architecture and Volta NVIDIA GPUs.
- Data for $L = 24, 48$ required respectively $\sim 270, 620$ hours on on the CINECA Marconi100 accelerated cluster.

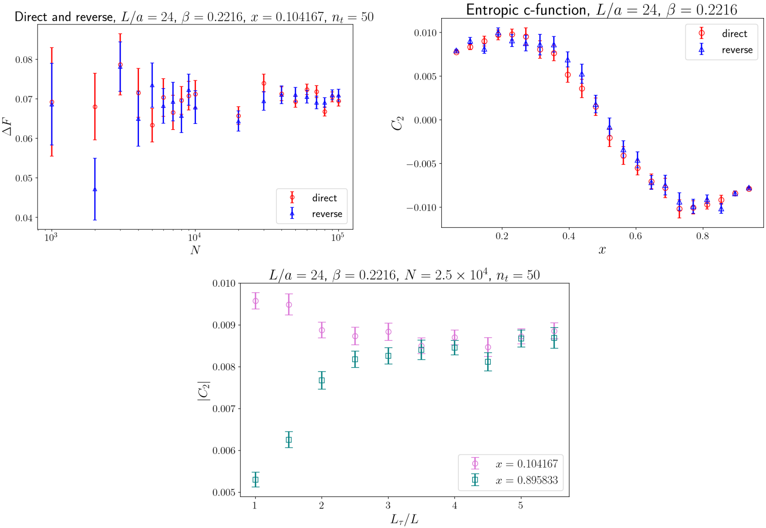
Benchmarks of the algorithm: $2D$







Benchmarks of the algorithm: $3D$



Duality transformation in $2D$