

SU(2) gauge fields and quantum entanglement

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Based mainly on:

- J. M. and T. Trześniewski,
Gauge fields and quantum entanglement,
Phys. Lett. B **810** (2020), 135808
- G. Czelusta and J. M.,
Quantum simulations of a qubit of space,
Phys. Rev. D **103** (2021) no.4, 046001
- G. Czelusta and J. M.,
Quantum circuits for the Ising spin networks,
[arXiv:2304.03559 [gr-qc]].

Quantum Entanglement in High Energy Physics 2023

Gauss law and gauge transformations

In electrodynamics (U(1)) the Gauss law is a secondary constraint of the theory, which generates gauge transformations.

One can notice that the Lagrangian: $\mu, \nu = 0, 1, 2, 3$

$$S = \int d^4x \mathcal{L} = -\frac{1}{4} \int d^4x F^{\mu\nu} F_{\mu\nu}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and A_μ is the four-potential (gauge field).

is invariant with respect to the gauge transformation:

$$A_\mu \rightarrow A'_\mu = U^\dagger A_\mu U - iU^\dagger \partial_\mu U$$

where $U = e^{i\lambda(x^\mu)} \in U(1)$ and $\lambda \in \mathbb{R}$, which leads to

$$A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu \lambda$$

The canonical momenta are:

$$\pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0 \quad (\text{primary constraint})$$
$$\pi^a = \frac{\partial \mathcal{L}}{\partial \dot{A}_a} = -E^a \quad a, b = 1, 2, 3$$

A_0 is a non-dynamical variable (Lagrange multiplier).

Employing the above, the ED Hamiltonian writes as:

$$H = \int d^3x (\pi^\mu \dot{A}_\mu - \mathcal{L}) = \frac{1}{2} \int d^3x (\vec{E}^2 + \vec{B}^2) + \int d^3x A^0 \partial_a E^a$$

From here, we find

$$0 = \dot{\pi}_0 = \{\pi_0, H\} = \partial_a E^a := C(\vec{E}) \quad (\text{secondary constraint})$$

which is the Gauss law:

$$\vec{\nabla} \cdot \vec{E} = 0$$

(or Gauss constraint)

One can consider the **smeared Gauss constraint**:

$$C[\lambda] := \int d^3x \lambda(x) C(\vec{E})$$

which forms the **first class algebra**:

$$\{C[\lambda_1], C[\lambda_2]\} = 0$$

and, therefore, the **Gauss constraint is a generator of the underlying symmetry**.
Indeed, because:

$$\delta A_a = \{A_a, C[\lambda]\} = \partial_a \lambda$$

we find that the Gauss constraint generates the **residual U(1) gauge symmetry**:

$$A_a \rightarrow A'_a = A_a + \delta A_a = A_a + \partial_a \lambda$$

SU(2) gauge field

Let us consider the **SU(2) connection** (1-form):

$$A = A_a^i \tau_i dx^a$$

The τ_i are generators of the $\mathfrak{su}(2)$ algebra $[\tau_i, \tau_j] = \epsilon_{ijk} \tau_k$

The SU(2) connection field is canonically conjugated to the **SU(2)** „electric” field $\vec{E}_a = E_a^i \tau_i$, where the arrow corresponds to the **internal (SU(2)) space**.

The canonical pair satisfies the following bracket:

$$\left\{ E_i^a(x), A_b^j(y) \right\} = \delta_b^a \delta_i^j \delta^{(3)}(x - y)$$

where $i, j = 1, 2, 3$ and $a, b = 1, 2, 3$

For theories invariant with respect to the local $SU(2)$ transformations (e.g. Yang-Mills theory, GR in the Ashtekar formalism):

$$A_a \rightarrow A'_a = U^\dagger A_a U + U^\dagger \partial_a U$$

where $U \in SU(2)$, the Gauss constraint takes the following form:

$$C_i := D_a E_i^a = \partial_a E_i^a + \epsilon_{ij}^{k} A_a^j E_k^a = 0$$

The smeared $SU(2)$ Gauss constraint is

$$C[\vec{\lambda}] := \int d^3x \lambda^i(x) C_i(E, A)$$

which satisfies the first class algebra:

$$\left\{ C[\vec{\lambda}_1], C[\vec{\lambda}_2] \right\} = C[[\vec{\lambda}_1, \vec{\lambda}_2]]$$

Holonomies - field theoretical viewpoint

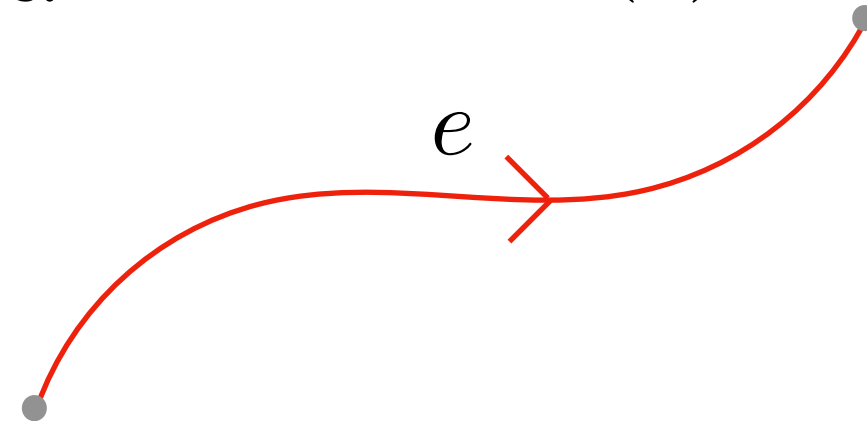
of the $SU(2)$ connection along a path e
are non-local objects defined as follows:

$$h_e[A] := \mathcal{P} \exp \int_e A$$

path $e : [0, 1] \rightarrow \Sigma$

$e(0) = s$ - source

$e(1) = t$ - target



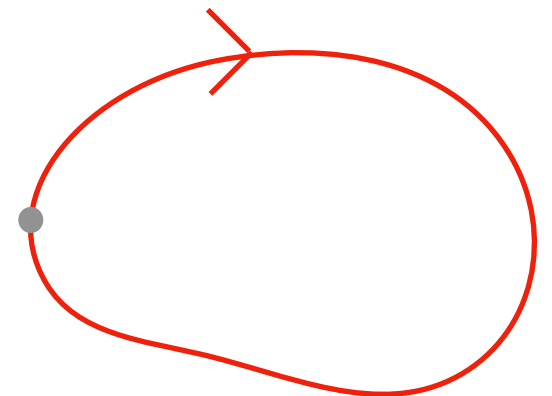
Under the gauge transformation the holonomy transforms as:

$$h_e[A] \rightarrow h'_e[A] = U^\dagger(e(0)) h_e[A] U(e(1)) = U_s^\dagger h_e[A] U_t$$

where $U_s := U(e(0))$ and $U_t := U(e(1))$

Gauge invariant objects
- Wilson loops:

$$W_e[A] := \text{tr}(h_e[A])$$



Holonomy-flux algebra

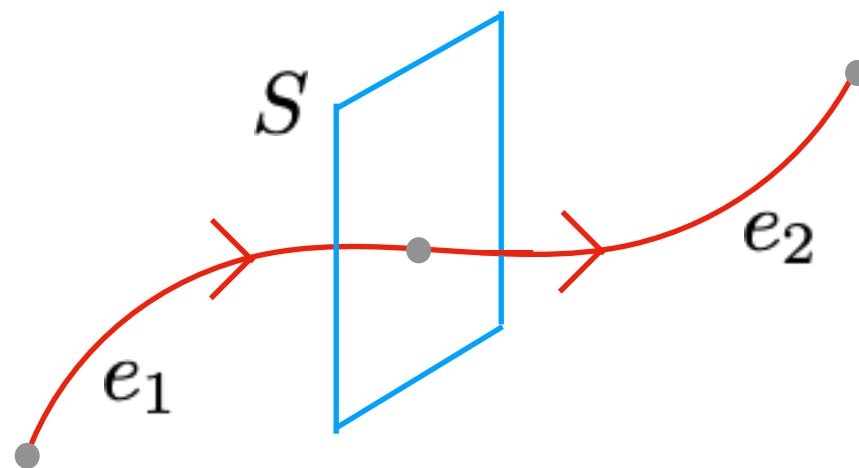
One can introduce **flux of the SU(2) „electric” field** through a 2-surface S :

$$\vec{F}_S[E] := \int_S \epsilon_{abc} E_i^a \tau^i dx^b \wedge dx^c$$

which satisfies the **holonomy-flux algebra**:

$$\{F_S^i[E], h_e[A]\} = -\iota(e, S) h_{e_1}[A] \tau^i h_{e_2}[A]$$

where $\iota(e, S) = \pm 1, 0$ is the **intersection number** and $e = e_1 \cup e_2$.



Holonomies - quantum mechanical viewpoint

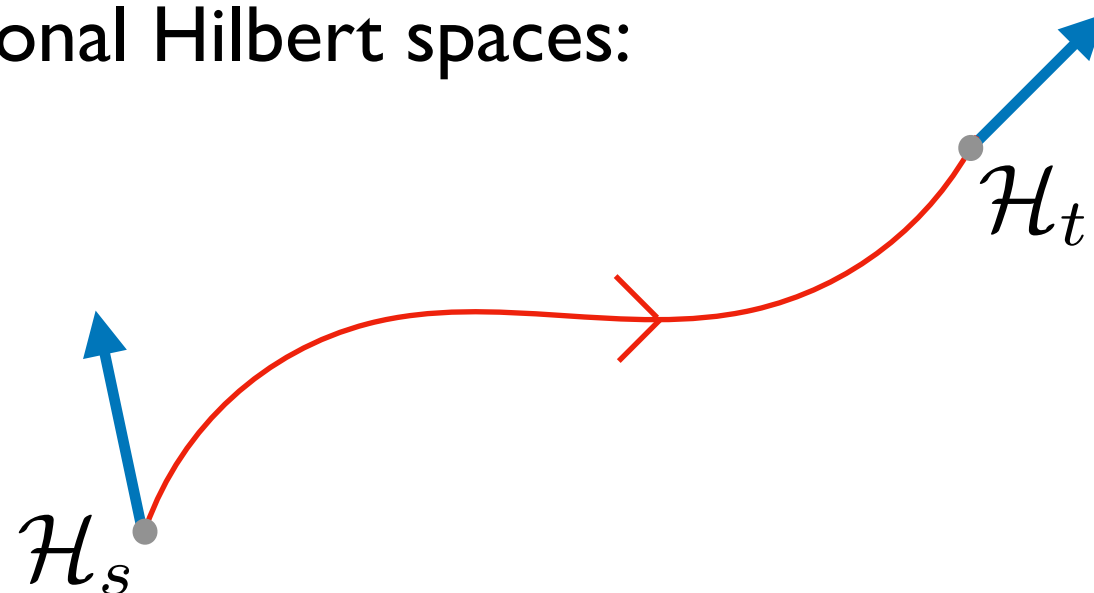
In the **fundamental ($j=1/2$) representation of $SU(2)$** , the holonomies are 2×2 $SU(2)$ matrices, which belong to the automorphism group of $\mathbb{C}^2$ (i.e. the space of non-relativistic spinors).

$\mathbb{C}^2$ equipped with the natural scalar product becomes the Hilbert space of a *qubit* system. Pure quantum states correspond to rays in $\mathbb{C}^2$.

The $SU(2)$ holonomy becomes then an isomorphism (unitary map) between the two 2-dimensional Hilbert spaces:

$$\mathcal{H}_s = \text{span} \{ |0\rangle_s, |1\rangle_s \}$$

$$\mathcal{H}_t = \text{span} \{ |0\rangle_t, |1\rangle_t \}$$



For general j -representation of $SU(2)$, the holonomies are $(2j+1) \times (2j+1)$ $SU(2)$ matrices, such that $\dim \mathcal{H}_s = 2j + 1 = \dim \mathcal{H}_t$

Holonomy as a unitary map

Employing the basis elements of the source and target Hilbert spaces, it is convenient to express an arbitrary holonomy map as:

$$h = h_{IJ} |I\rangle_{st} \langle J| \in \mathcal{H}_s \otimes \mathcal{H}_t^* \quad \text{where } I, J = 0, 1$$

The action of this unitary map can be either **left-handed** or **right-handed**:

$$h_L : \mathcal{H}_s^* \rightarrow \mathcal{H}_t^* \quad h_R : \mathcal{H}_t \rightarrow \mathcal{H}_s$$

The Hermitian conjugation of h: $h^\dagger = h_{IJ}^* |J\rangle_{ts} \langle I| \in \mathcal{H}_t \otimes \mathcal{H}_s^*$

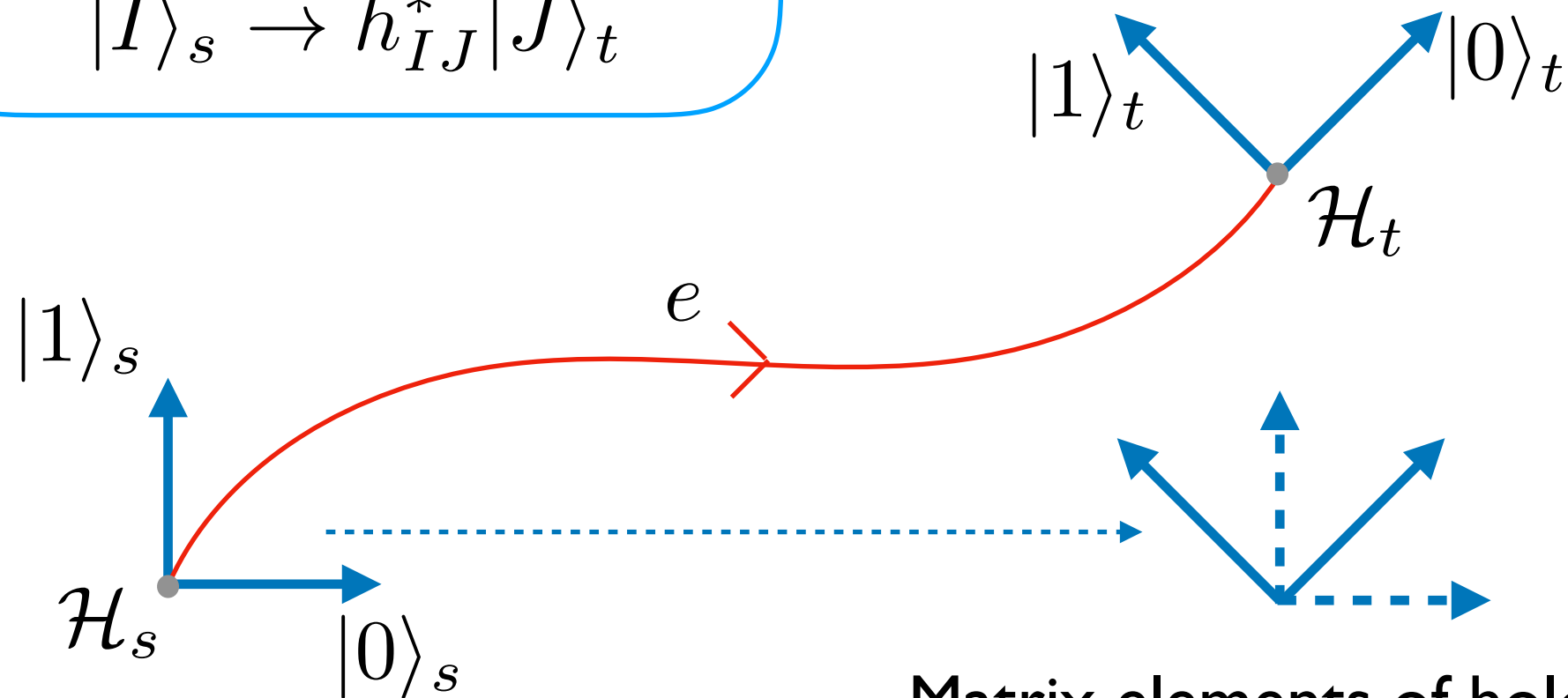
acts as $h_L^\dagger : \mathcal{H}_t^* \rightarrow \mathcal{H}_s^*$ and $h_R^\dagger : \mathcal{H}_s \rightarrow \mathcal{H}_t$

Example: A basis state ${}_s \langle K| \in \mathcal{H}_s^*$ at the point s is mapped into:

$${}_s \langle K| h = h_{IJ} {}_s \langle K| I \rangle_{st} \langle J| = h_{KJ} \langle J| \in \mathcal{H}_t^* \quad \text{at the point t.}$$

Mapping of the basis states:

$$|I\rangle_s \rightarrow h_{IJ}^* |J\rangle_t$$

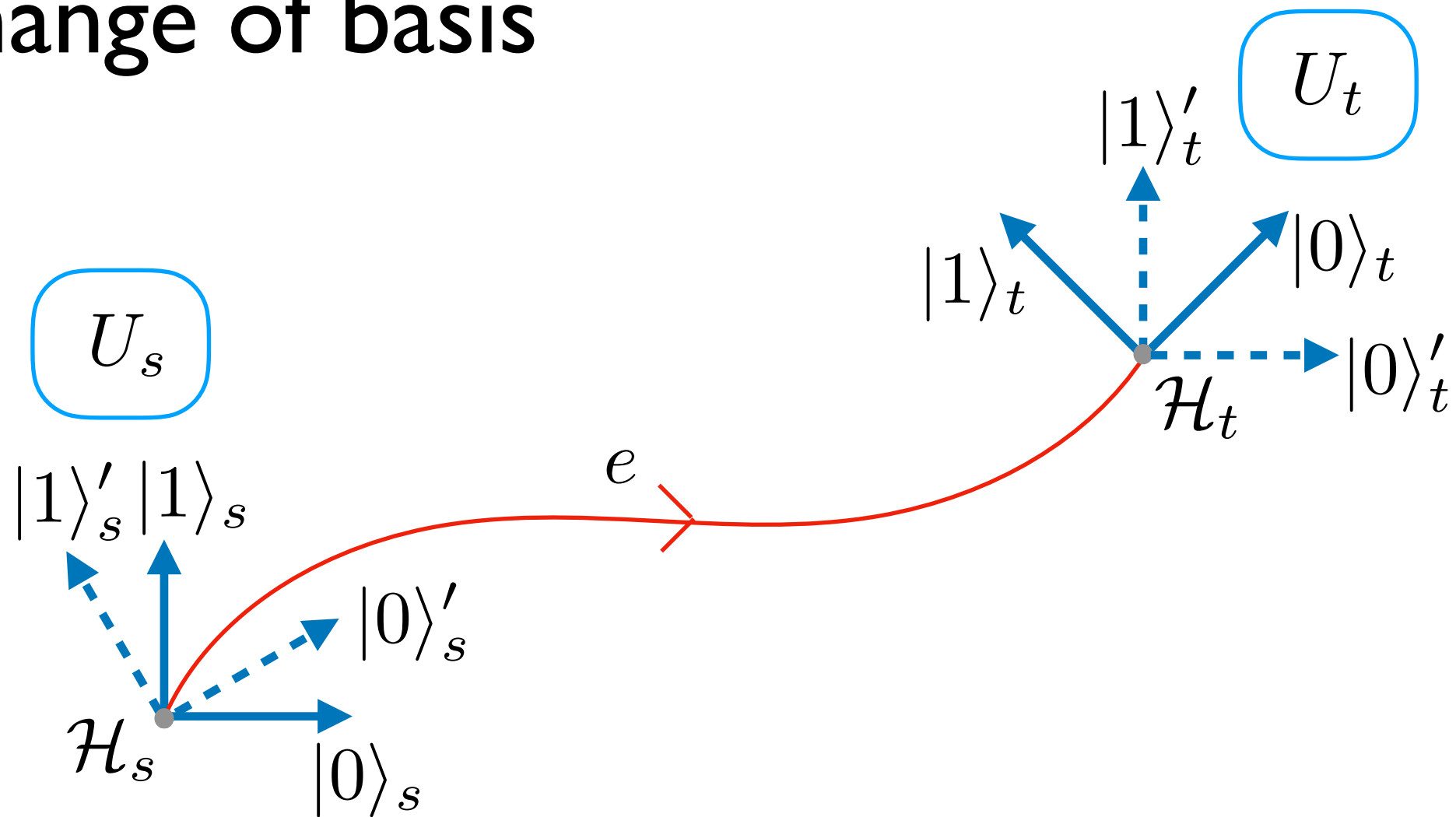


Matrix elements of holonomies
are the coefficients of decomposition
of one basis into the other

Physical interpretation: gravitational holonomies
describe displacement of a quantum system from point
 s to point t in the gauge field:

$$|\psi\rangle \rightarrow \mathcal{P}e^{\int_e A} |\psi\rangle$$

Change of basis

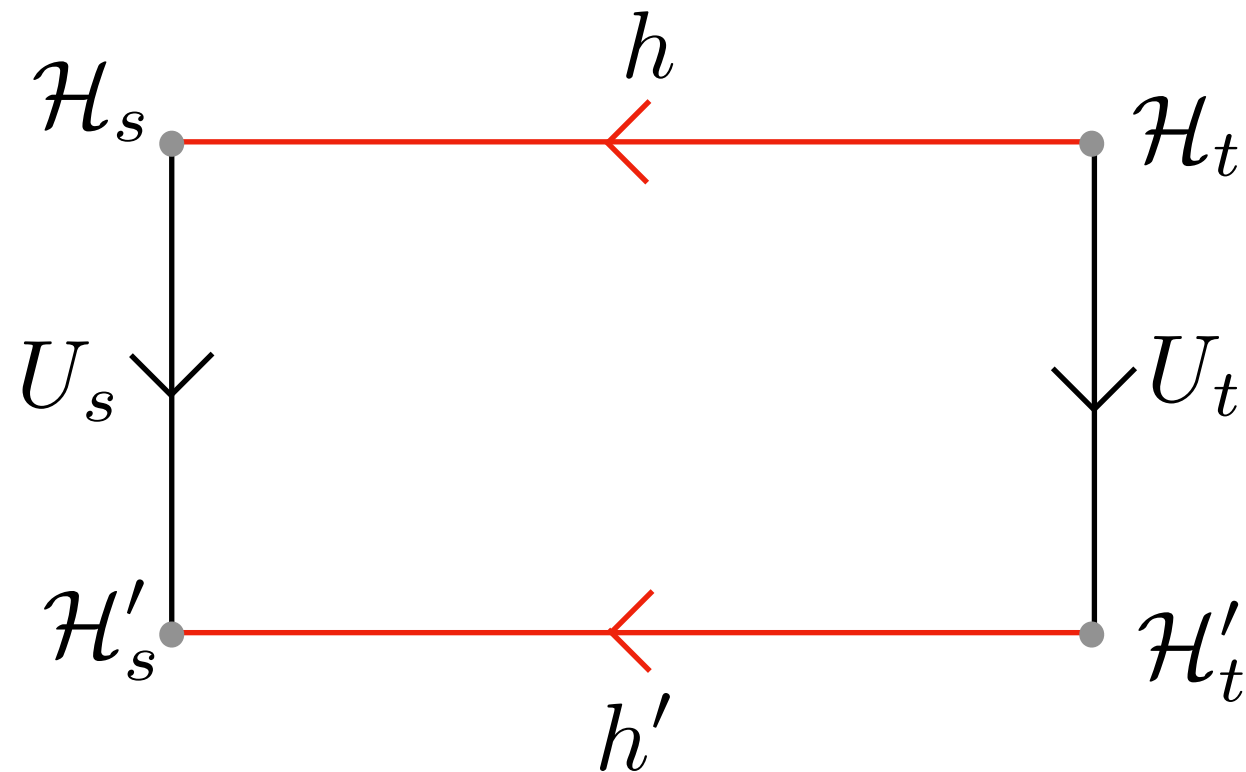


Physics does not depend on the choice of basis.

One can perform unitary transformation: $|I\rangle' = U|I\rangle$

or using components of the unitary matrix $|I\rangle' = U_{JI}|J\rangle$

How the holonomies transform under the change of bases?



The action of holonomy is preserved under the transformation of bases **if**

$$h'_{IJ} |I\rangle'_{st} \langle J|' = U_{s,KI} h'_{IJ} U_{t,JL}^\dagger |K\rangle_{st} \langle L| = h_{IJ} |I\rangle_{st} \langle J|$$

which leads to the transformation rule:

$$h \rightarrow h' = U_s^\dagger h U_t$$

It is clear that the change of h under unitary transformations in the source and target spaces is equivalent to the action of a SU(2) gauge transformation.

Holonomies as wave functions

Functions of holonomies, equipped with Haar measure on $SU(2)$ Lie group form a Hilbert space:

$$\mathcal{H} = L^2(SU(2))$$

so that: $\varphi(h_e) \in L^2(SU(2))$. Following the [Peter-Weyl theorem](#):

$$L^2(SU(2)) = \oplus_j (\mathcal{H}_j \otimes \mathcal{H}_j^*)$$

where $\mathcal{H}_j$ is a spin- j Hilbert space.

The [orthonormal basis states](#) in the Hilbert space, for a given path e , are:

$$\varphi(h_e)^j = \frac{1}{\sqrt{2j+1}} (h_{IJ})^j |I\rangle_{st} \langle J| \in \mathcal{H}_{j,s} \otimes \mathcal{H}_{j,t}^* \quad I, J = 0, \dots, 2j$$

For spin-1/2: $\varphi(h_e)^{1/2} = \frac{1}{\sqrt{2}} h_{IJ} |I\rangle_{st} \langle J| \in \mathcal{H}_{1/2} \otimes \mathcal{H}_{1/2}^* \quad I, J = 0, 1$

States in $\mathcal{H}_s \otimes \mathcal{H}_t$

What are the states in $\mathcal{H}_s \otimes \mathcal{H}_t$?

Because of the isomorphism between $\mathcal{H}_j$ and $\mathcal{H}_j^*$ **one can map states from $\mathcal{H}_s \otimes \mathcal{H}_t^*$ to states in $\mathcal{H}_s \otimes \mathcal{H}_t$.**

In particular, for spin-1/2:

$$\Psi(h_e) \in \mathcal{H}_{1/2} \otimes \mathcal{H}_{1/2}^* \rightarrow |\Psi\rangle = \frac{1}{\sqrt{2}} h_{IJ}^* |I\rangle_s |J\rangle_t \in \mathcal{H}_{1/2} \otimes \mathcal{H}_{1/2}$$

where h_{IJ} are matrix components of the SU(2) matrix.

This way of expressing 2-particle states is known in quantum information community (e.g. Bengtsson & Życzkowski, 2017)

The state can be used to introduce anti-linear map & relation to quantum teleportation (Czech, Lamprou & Susskind, 2018; Czech, De Boer, Ge & Lamprou, 2019)

Improved analysis, gravity, networks, etc. (JM & Trześniewski, 2020)

The state $|\Psi\rangle := \frac{1}{\sqrt{2}} h_{IJ}^* |I\rangle_s |J\rangle_t \in \mathcal{H}_s \otimes \mathcal{H}_t$

is a **maximally entangled** state. The density matrix:

$$\hat{\rho} = |\Psi\rangle\langle\Psi| = \frac{h_{IJ}^* h_{KL}}{2} (|I\rangle_s \langle K|) (|J\rangle_t \langle L|)$$

Unitarity of h:

The reduced density matrix:

$$h_{IK} h_{KJ}^\dagger = \delta_{IJ}$$

$$\hat{\rho}_s := \text{tr}_t(\hat{\rho}) = \frac{h_{IJ}^* h_{JK}^T}{2} (|I\rangle_s \langle K|) = \frac{1}{2} \hat{I}$$

The same for
 $\hat{\rho}_t := \text{tr}_s(\hat{\rho})$

The mutual information is maximal:

$$I(s : t) = S(\rho_s) + S(\rho_t) - S(\rho) = 2 \ln 2$$

$$S(\rho_{s,t}) = -\text{tr}(\rho_{s,t} \ln \rho_{s,t}) = \ln 2$$

$$S(\rho) = 0 \text{ (pure state)}$$

Antilinear map

Equivalently to the case of holonomy, one can the following map:

$$\mathcal{H}_s^* \ni {}_s\langle I| \rightarrow \left[\sqrt{2} |\Psi\rangle \circ C \right] ({}_s\langle I|) = h_{IJ}^* |J\rangle_t \in \mathcal{H}_t$$

where C is the complex conjugation operation.

Change of bases leads to

$$\begin{aligned} {}_s\langle I|' &\rightarrow \left[\sqrt{2} |\Psi\rangle \circ C \right] ({}_s\langle I|') = \sqrt{2} (U_{s,JI}^*)^* {}_s\langle J|\Psi\rangle \\ &= U_{s,IJ}^T h_{JL}^* |L\rangle_t = U_{s,IJ}^T h_{JL}^* U_{t,LM}^* |M\rangle_t' \end{aligned}$$

which leads to the following transformation rule:

$$h_{JM} \rightarrow h'_{JM} = U_{s,IJ}^\dagger h_{JL} U_{t,LM}$$

The map is equivalent to the $SU(2)$ gauge transformation.

Gauge transformations from entanglement

Every maximally entangled 2-particle state can be associated with SU(2) holonomy.

For qubits:

$$|\Psi\rangle = \frac{1}{\sqrt{2}} h_{IJ}^* |I\rangle_s |J\rangle_t \in \mathcal{H}_s \otimes \mathcal{H}_t \quad \text{where} \quad I, J = 0, 1$$

In general:

$$|\Psi\rangle = \frac{1}{\sqrt{2j+1}} h_{IJ}^* |I\rangle_s |J\rangle_t \quad \text{where} \quad I, J = 0, \dots, 2j$$

The h_{IJ} are components of SU(2) matrix, which under the change of bases transform as $h_{JM} \rightarrow h'_{JM} = U_{s,IJ}^\dagger h_{JL} U_{t,LM}$

This is possible because $|\Psi\rangle$ defines the unitary map.

Example: Maximally entangled singlet state:

$$|\Psi\rangle = |\Phi^-\rangle := \frac{1}{\sqrt{2}} (|0\rangle_s |1\rangle_t - |1\rangle_s |0\rangle_t)$$

The state can be written as

$$|\Psi\rangle = \frac{1}{\sqrt{2}} h_{IJ}^* |I\rangle_s |J\rangle_t \in \mathcal{H}_s \otimes \mathcal{H}_t$$

where $h = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma_y = e^{-i\frac{\pi}{2}\sigma_y}$

For simplicity, let us consider a one-parameter family of rotations:

$$U(\theta) = e^{-i\theta\sigma_y} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

The angle changes continuously between s and t , such that:

$$\theta_s := \theta(e(0) = s) \quad \theta_t := \theta(e(1) = t)$$

Under the change of bases, the map h transforms as follows

$$\begin{aligned} h \rightarrow h' &= U_s^\dagger h U_t = e^{-i\frac{\pi}{2}\sigma_y - i(\theta_t - \theta_s)\sigma_y} \\ &= h e^{-i\sigma_y \int_e \partial_a \theta dx^a} \end{aligned}$$

One can, therefore, conclude that the matrix h can be written as exponentiation of a certain integral, such that its integrand under the change of bases transforms as:

$$\text{integrand} \rightarrow \text{integrand} + \partial_a \theta$$

The integrand is what we can call a gauge field!

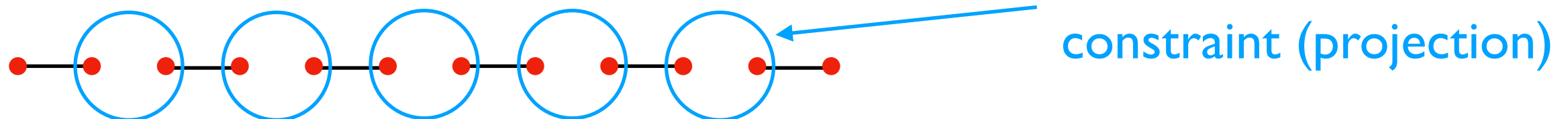
The $U(1)$ gauge symmetry is a residue of the $SU(2)$ gauge symmetry restricted to the case of a one-parameter family of transformations.

How to understand this?

Let us consider a discrete system of quantum DoF, e.g a spin chain



The DoF form maximally entangled pairs, e.g.



Physical quantities have to be unaffected by local change of bases

The change of bases lead to the characteristics transformation law of the holonomies representing the maximally entangled pairs. Transformations laws of the local DoF must compensate the factors resulting from the change of bases (precursors of gauge transformations).

In the continuous limit, the field theoretical description shall be recovered.

CONJECTURE: The gauge field theories are continuous approximations to discrete quantum system characterized by maximally entangled pairs.

Constructing discrete (lattice) SU(2) gauge theory

... employing the holonomies and fluxes.

Let us consider the **L links** e , which meet at **N nodes**.

The full quantum holonomy-flux algebra (between holonomies and conjugated fluxes) is:

$$[F_e^j, h_{e'}] = i\delta_{ee'} h_{e'} \tau^j$$

$$[h_e, h_{e'}] = 0$$

$$[F_e^i, F_{e'}^j] = i\delta_{ee'} \epsilon^{ij}_k F_e^k$$

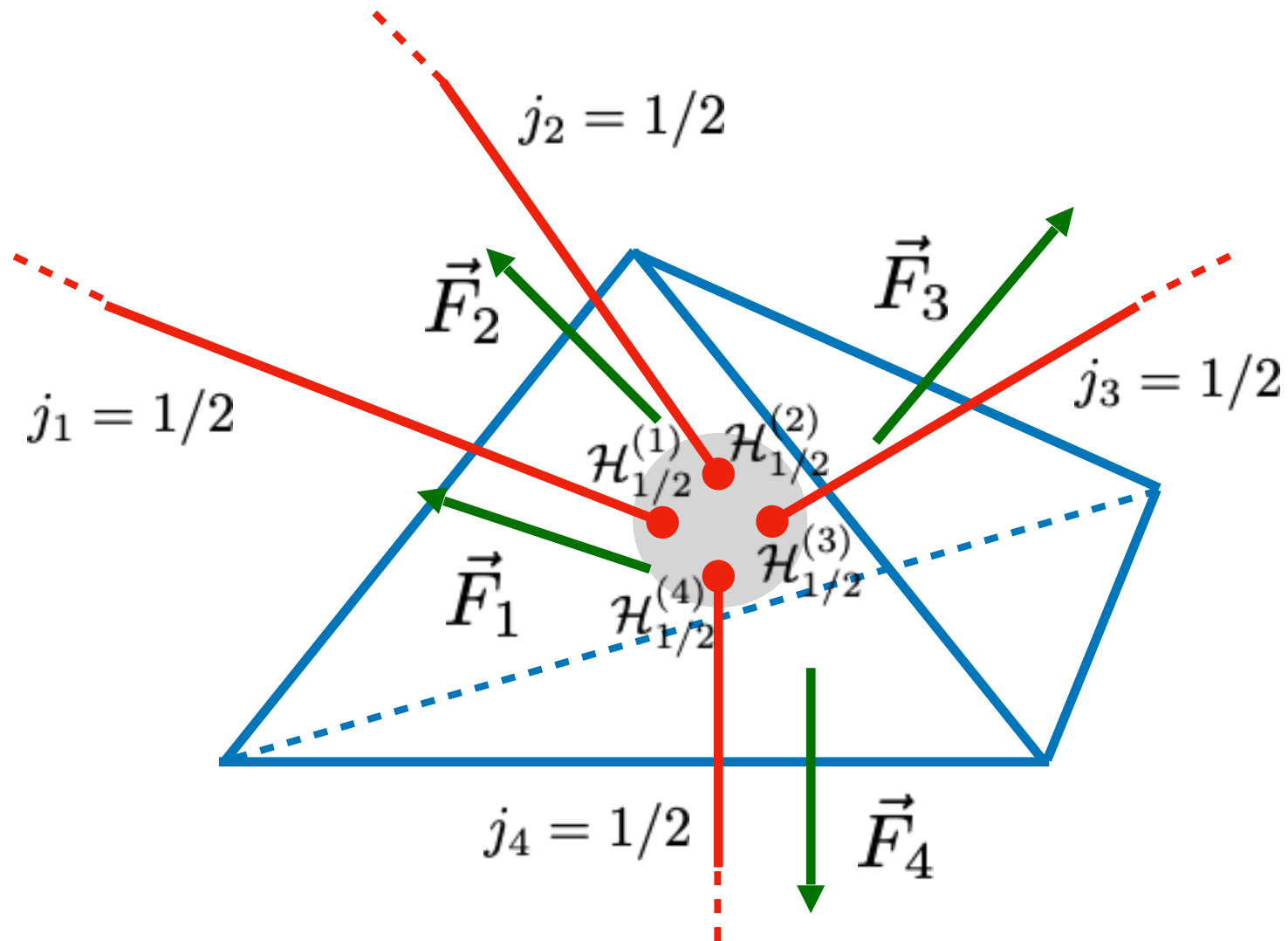
Therefore, **the conjugated fluxes are just angular momenta**: $\vec{F} = \vec{J}$

And the **kinematical Hilbert space** is:

$$L^2(SU(2)^L)$$

Imposing the Gauss constraint

For simplicity, let us consider 4-valent nodes and fundamental (spin-1/2) representations at the links.



$\hat{P}_G$ - Projection onto spin-0 subspace by the virtue of the Gauss constraint.

The Gauss constraint tells us that the four fluxes (angular momenta) conjugated to the holonomies sum-up to zero:

$$\sum_{i=1}^4 \vec{F}_i = 0$$

Equation of a tetrahedron with the areas of the faces:

$$A_i = ||\vec{F}_i||$$

Quantum tetrahedron

A state of the quantum tetrahedron is:

$$|\Psi\rangle \in \mathcal{H}_{j_1} \otimes \mathcal{H}_{j_1} \otimes \mathcal{H}_{j_3} \otimes \mathcal{H}_{j_4} \quad \text{such that} \quad (\sum_{a=1}^4 \hat{\vec{J}}_a) |\Psi\rangle = 0$$

where $\mathcal{H}_{j_a} = \text{span}\{|j_a, -j_a\rangle, \dots, |j_a, j_a\rangle\}$

$$\text{and } \hat{\vec{J}}_a \cdot \hat{\vec{J}}_a |j_a, m_a\rangle = j_a(j_a + 1) |j_a, m_a\rangle$$

So, the $|\Psi\rangle$ state belongs to the **SU(2)-invariant subspace** of product of spins.

Please note that the states allows for quantum communication without a shared reference frame:

$$\hat{\rho} = \int_{SU(2)} dg \hat{U}(g)^4 \hat{\rho} \hat{U}^\dagger(g)^4$$

We call the subspace an **intertwiner space**:

$$|\mathcal{I}\rangle \in \text{Inv}(\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_1} \otimes \mathcal{H}_{j_3} \otimes \mathcal{H}_{j_4})$$

dim Inv = number of linearly independent singlet states

The special case $j_1 = j_2 = j_3 = j_4 = j$

$$\dim \text{Inv}(\mathcal{H}_j \otimes \mathcal{H}_j \otimes \mathcal{H}_j \otimes \mathcal{H}_j) = 2j + 1$$

In what follows we will focus on the fundamental representation of $SU(2)$: $j=1/2$

$$\dim \text{Inv}(\mathcal{H}_{1/2} \otimes \mathcal{H}_{1/2} \otimes \mathcal{H}_{1/2} \otimes \mathcal{H}_{1/2}) = 2$$

This comes from the fact that:

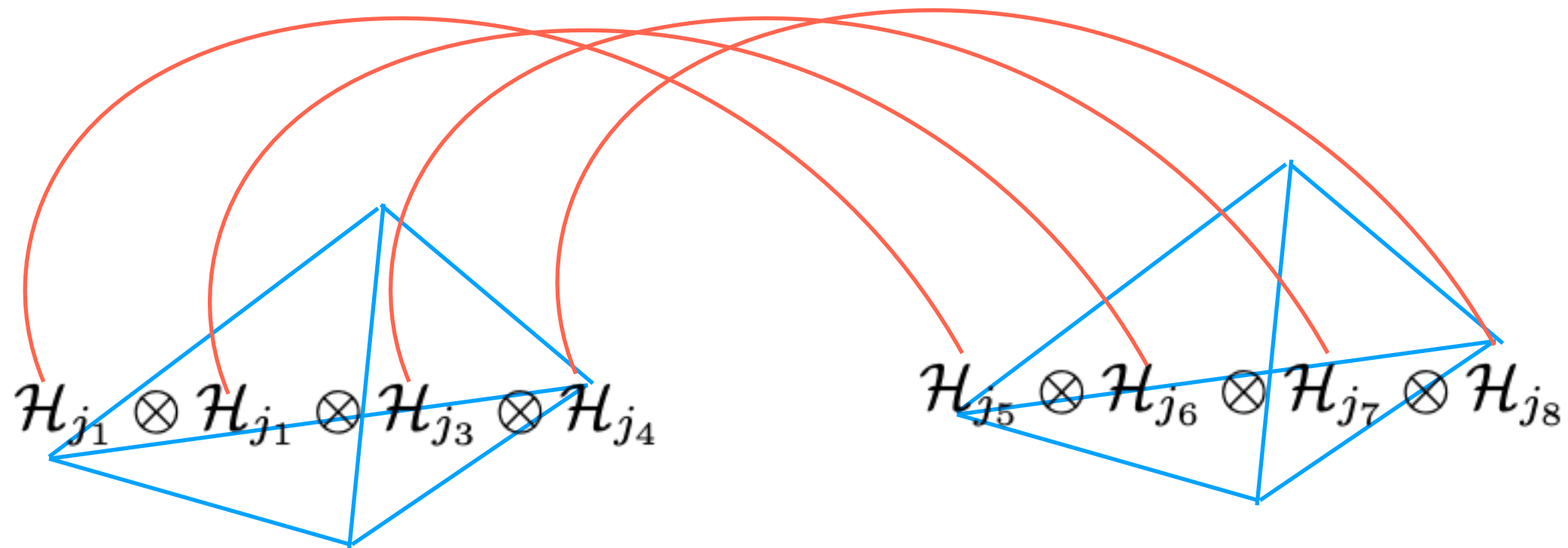
$$\begin{aligned} &\mathcal{H}_{1/2} \otimes \mathcal{H}_{1/2} \otimes \mathcal{H}_{1/2} \otimes \mathcal{H}_{1/2} \\ &= 2\mathcal{H}_0 \oplus 3\mathcal{H}_1 \oplus \mathcal{H}_2 \end{aligned}$$

The invariant subspace is two-dimensional - **intertwiner qubit**

Consequently, the **physical Hilbert space** reduces to:

$$L^2(SU(2)^L / SU(2)^N) = \bigotimes_{i=1}^N \mathcal{H}_{1/2}^{(i)}$$

Building gauge invariant states from holonomies



One can begin with the product spaces in which holonomies live, i.e.:

$$\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_5} \quad \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_6} \quad \mathcal{H}_{j_3} \otimes \mathcal{H}_{j_7} \quad \mathcal{H}_{j_4} \otimes \mathcal{H}_{j_8}$$

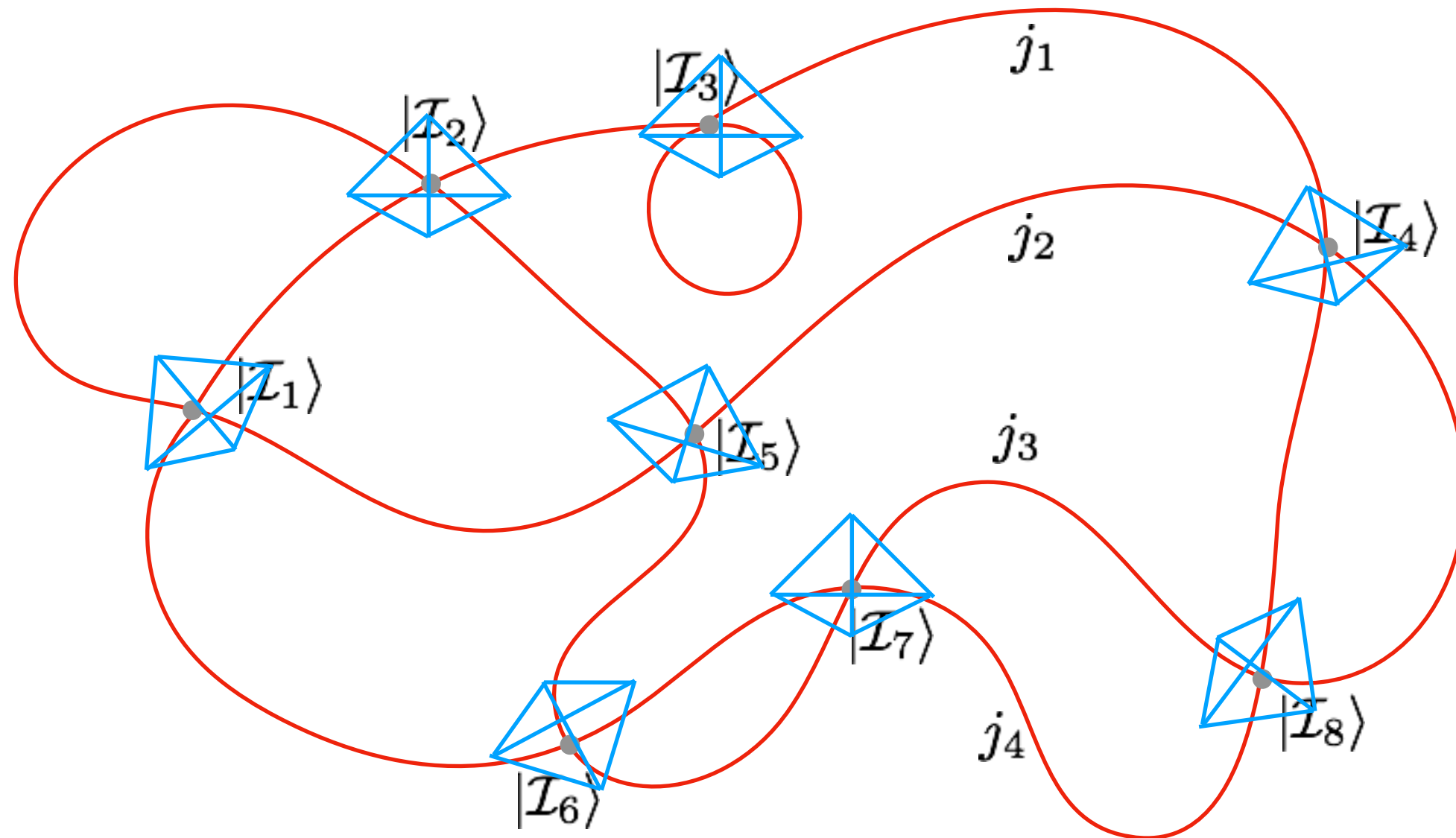
take their product $\bigotimes_{i=1}^8 \mathcal{H}_{j_i}$

and impose the Gauss constraint, which leads to:

$$\text{Inv}(\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3} \otimes \mathcal{H}_{j_4}) \otimes \text{Inv}(\mathcal{H}_{j_5} \otimes \mathcal{H}_{j_6} \otimes \mathcal{H}_{j_7} \otimes \mathcal{H}_{j_8})$$

This construction generalizes to an arbitrary number of nodes and leads to a concept of **spin networks**. (Penrose, Rovelli, Smolin, ...)

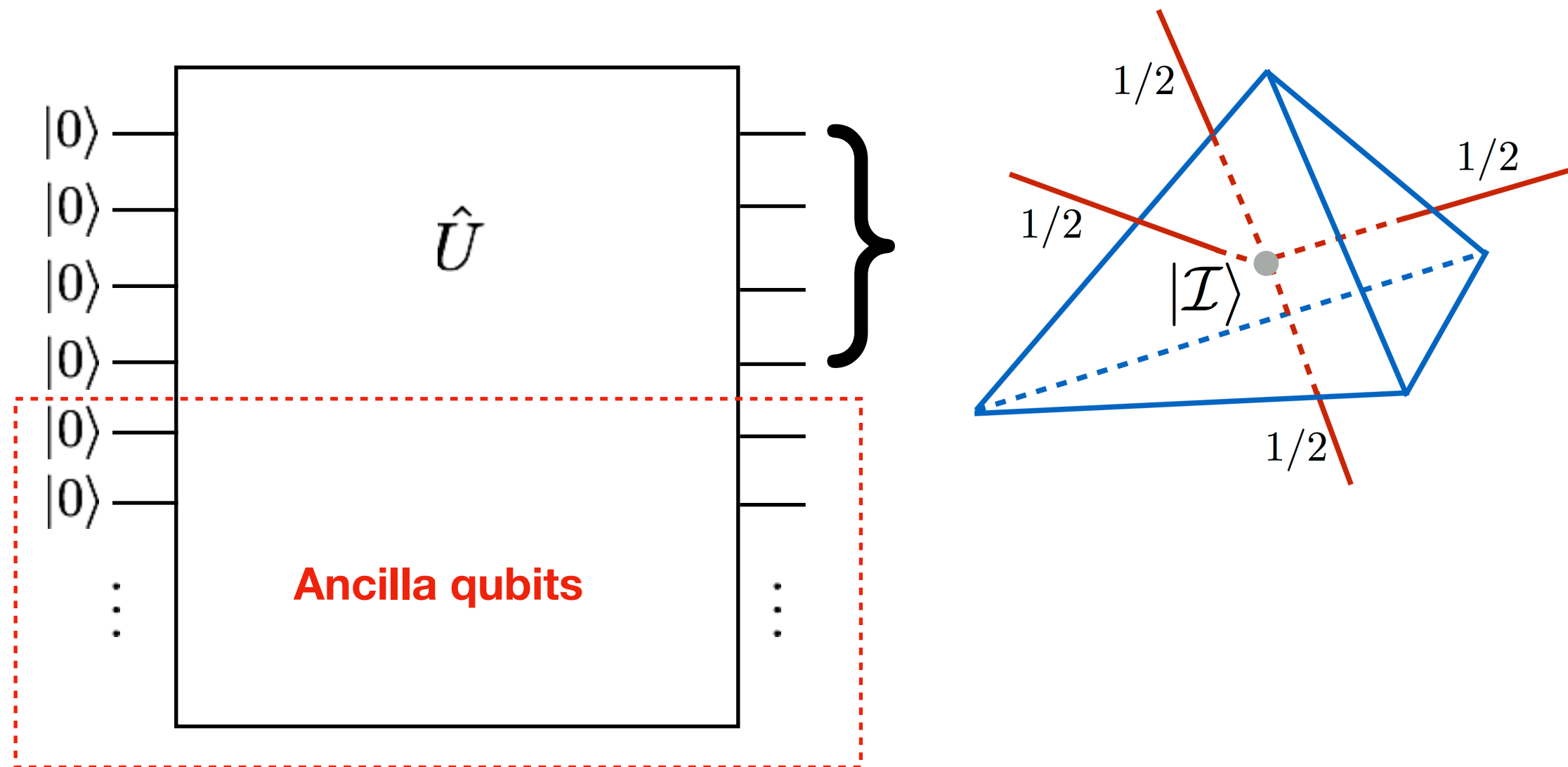
Spin networks - states of SU(2) gauge theory



Spin labels - irreducible representations of the SU(2) group: $j_i = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$

Local SU(2) gauge invariance (Gauss constraint) implies that **spins sum up to zero at the nodes** - degeneracy leads to **intertwiner spaces**.

Quantum circuit for the quantum tetrahedron



At least four qubits are needed to create a single **intertwiner qubit** state.

K. Li *et al.* (2019), JM (2019), G. Czelusta & JM (2021), L. Cohen *et al.* (2021)

A general intertwiner qubit state

$$|\mathcal{I}\rangle = \cos(\theta/2)|0_s\rangle + e^{i\phi} \sin(\theta/2)|1_s\rangle$$

Intertwiner qubit base states in the s-channel (four qubit singlets):

$$|0_s\rangle = |S\rangle \otimes |S\rangle,$$

$$|1_s\rangle = \frac{1}{\sqrt{3}} (|T_+\rangle \otimes |T_-\rangle + |T_-\rangle \otimes |T_+\rangle - |T_0\rangle \otimes |T_0\rangle).$$

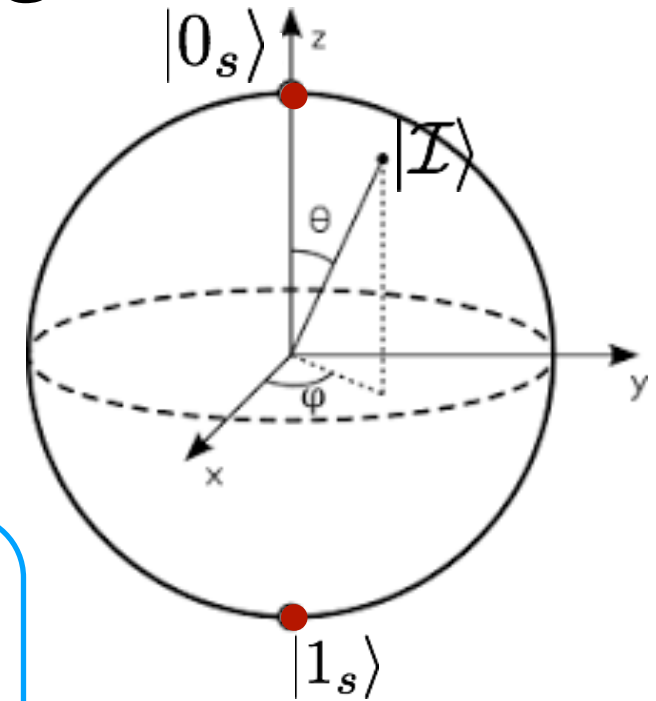
Singlet and triplet states for two spin-1/2 particles:

$$|S\rangle = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle),$$

$$|T_+\rangle = |00\rangle,$$

$$|T_0\rangle = \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle),$$

$$|T_-\rangle = |11\rangle.$$



Preparation of the state $|\mathcal{I}\rangle$

Task: Find a circuit U which generates a general intertwiner qubit state:

$$|\mathcal{I}\rangle = \hat{U}_{\mathcal{I}}|0000\rangle$$

The procedure is **not** unique!

$$|\mathcal{I}\rangle = \cos(\theta/2)|0_s\rangle + e^{i\phi} \sin(\theta/2)|1_s\rangle$$

can be expressed as:

$$\begin{aligned} |\mathcal{I}\rangle = & \frac{c_1}{\sqrt{2}}(|0011\rangle + |1100\rangle) \\ & + \frac{c_2}{\sqrt{2}}(|0101\rangle + |1010\rangle) \\ & + \frac{c_3}{\sqrt{2}}(|0110\rangle + |1001\rangle) \end{aligned}$$

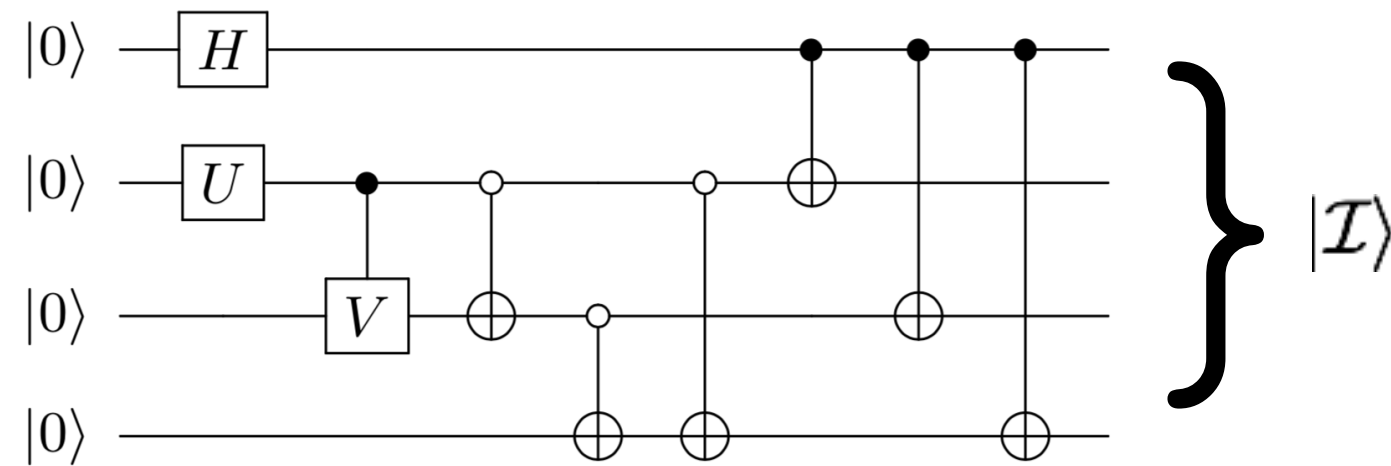
$$c_1 = \sqrt{\frac{2}{3}} e^{i\phi} \sin(\theta/2),$$

$$\begin{aligned} c_2 &= \frac{1}{\sqrt{2}} \left(\cos(\theta/2) - \frac{1}{\sqrt{3}} e^{i\phi} \sin(\theta/2) \right) \\ &= \frac{e^{i\chi_+}}{\sqrt{2}} \sqrt{1 - \frac{2}{3} \sin^2(\theta/2) - \frac{\sin \theta \cos \phi}{\sqrt{3}}}, \end{aligned}$$

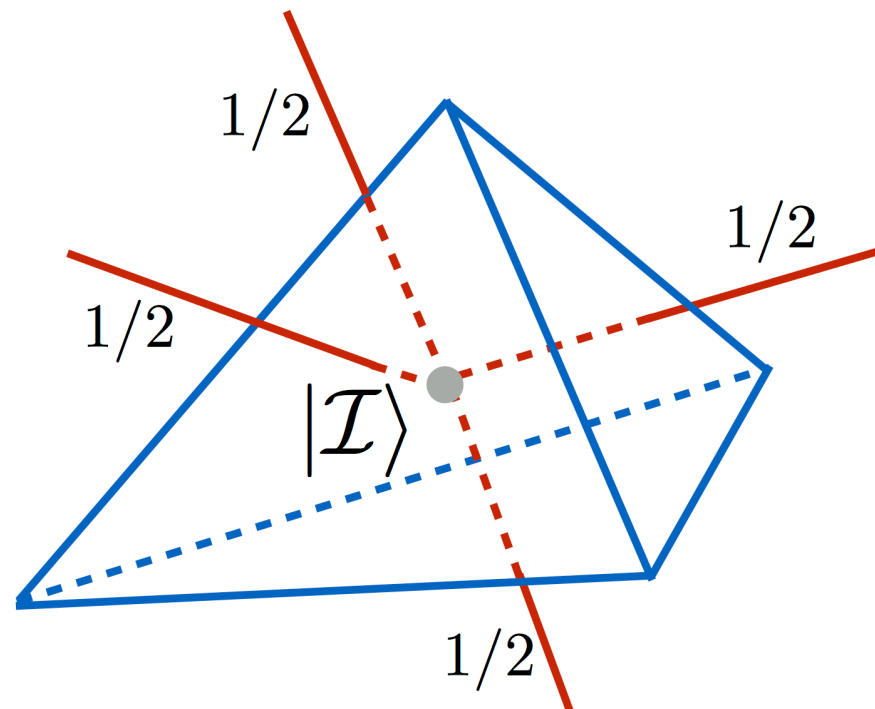
$$\begin{aligned} c_3 &= \frac{1}{\sqrt{2}} \left(-\cos(\theta/2) - \frac{1}{\sqrt{3}} e^{i\phi} \sin(\theta/2) \right) \\ &= \frac{e^{i\chi_-}}{\sqrt{2}} \sqrt{1 - \frac{2}{3} \sin^2(\theta/2) + \frac{\sin \theta \cos \phi}{\sqrt{3}}}, \end{aligned}$$

$$\sum_{i=1}^3 |c_i|^2 = 1 \quad \sum_{i=1}^3 c_i = 0$$

A quantum circuit for quantum tetrahedron:



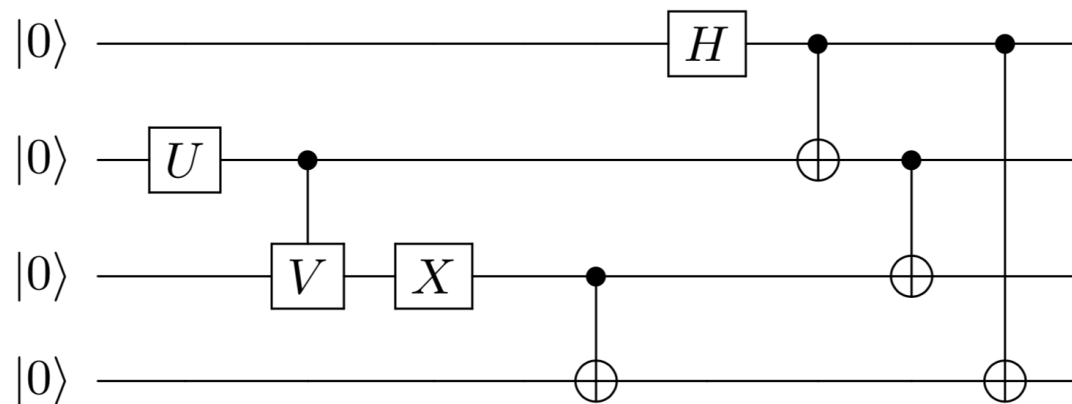
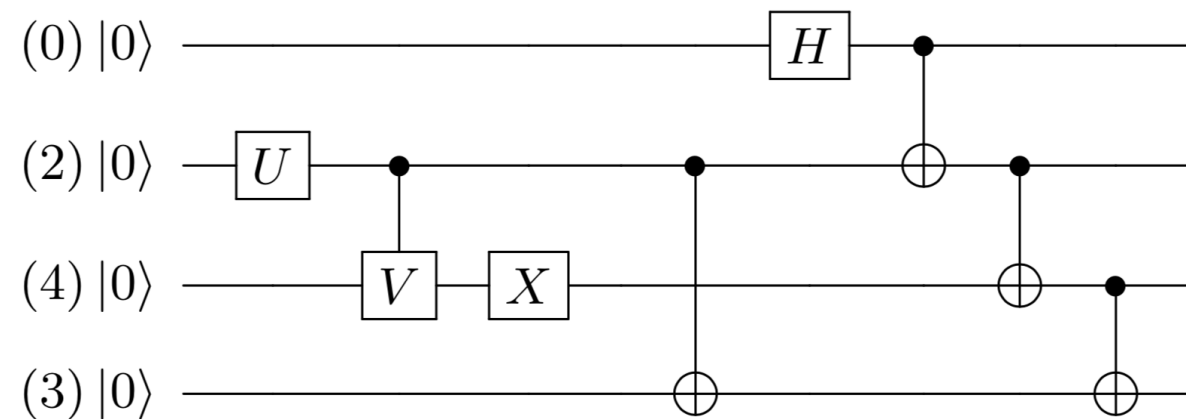
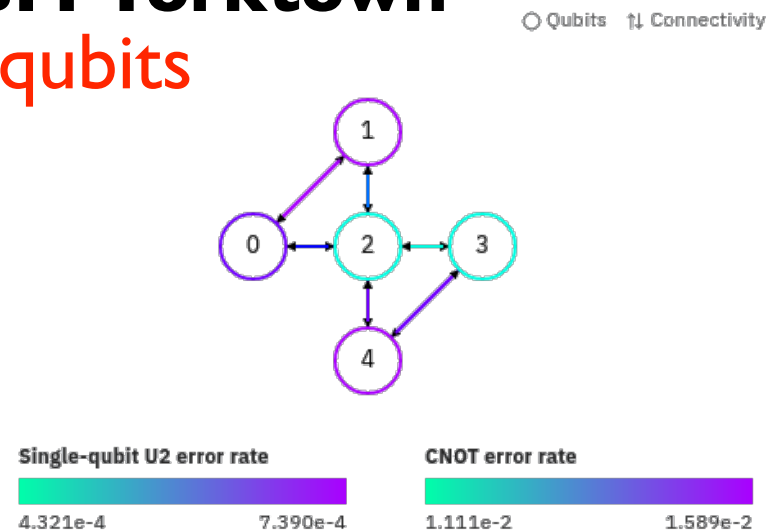
=



Transpilation

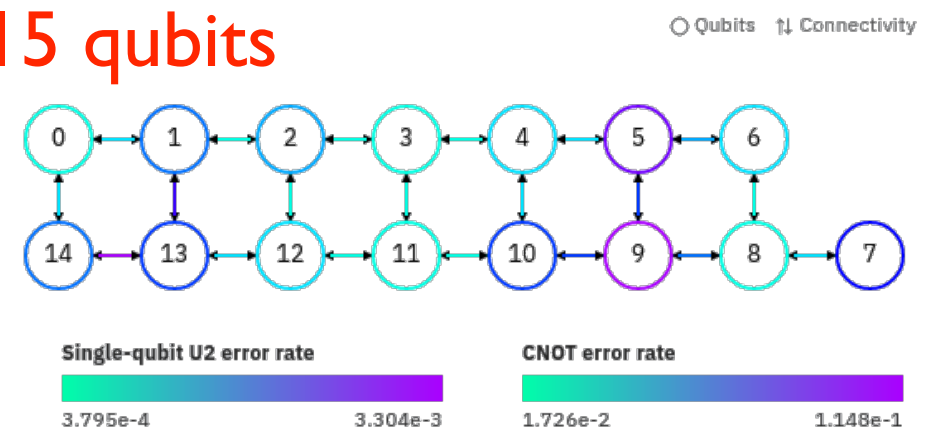
The circuit has to be fitted to the topology of a quantum processor:

IBM Yorktown 5 qubits

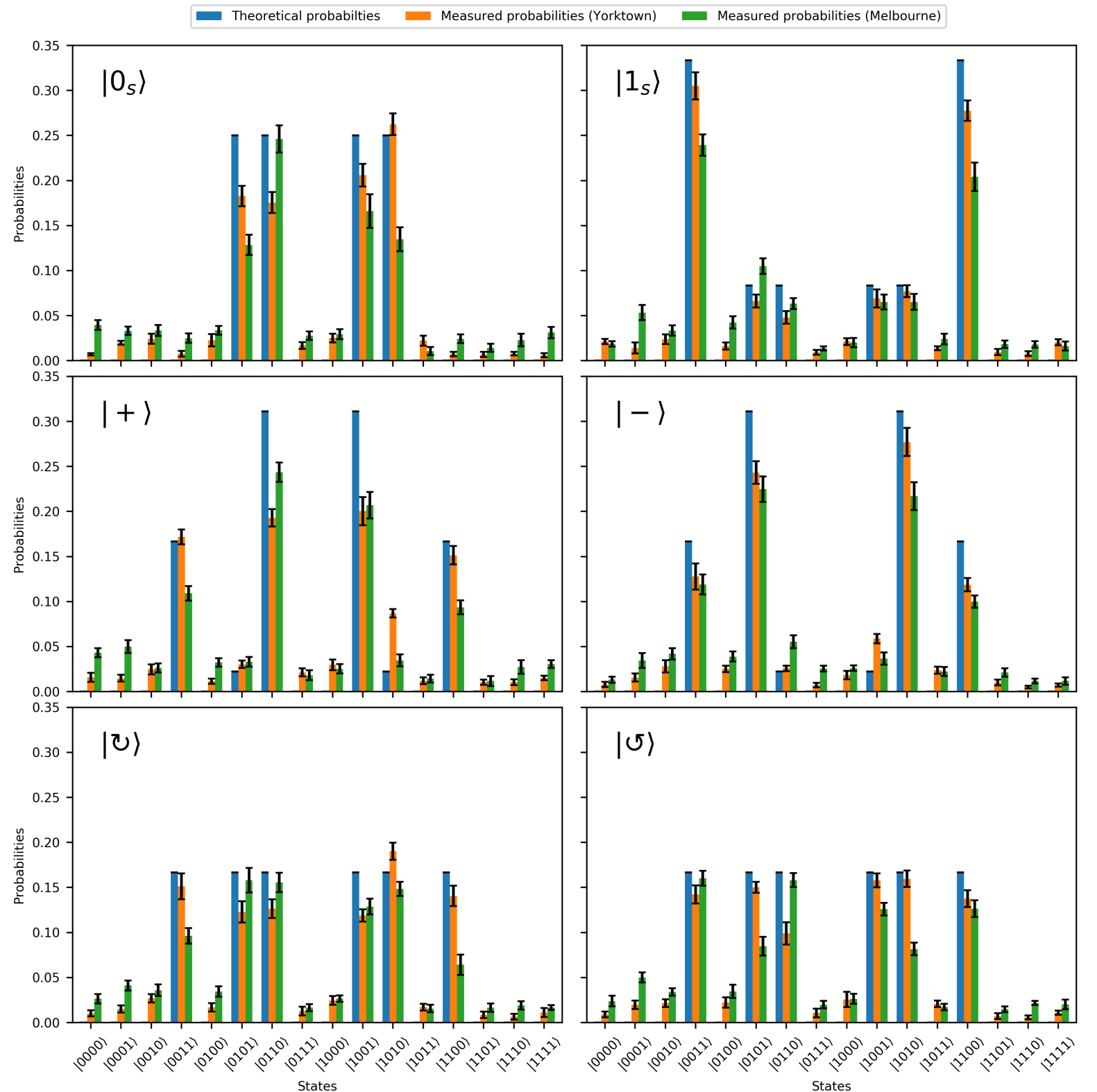
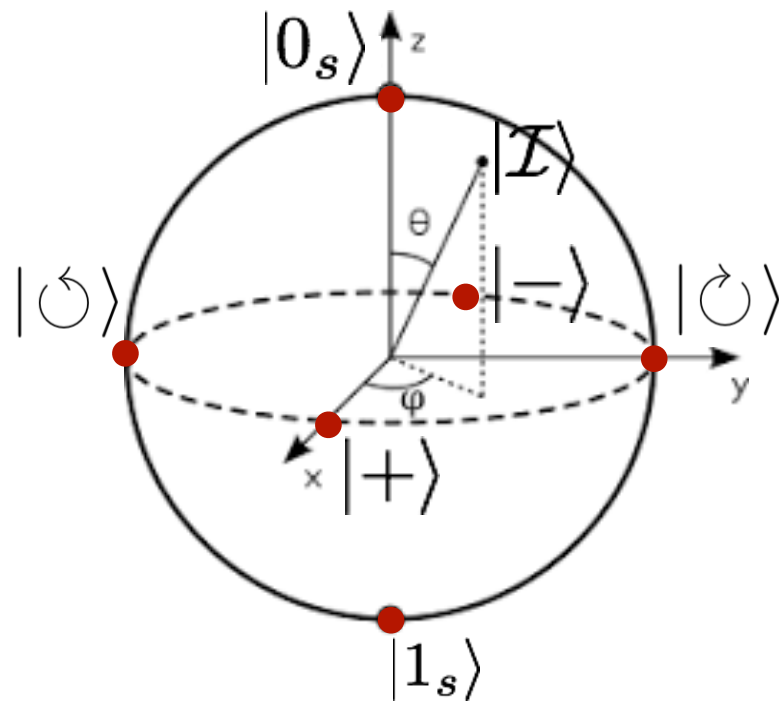


IBM Melbourne

15 qubits



Simulations



A sequence of 10 computational rounds each containing 1024 shots was performed for every of the considered states.

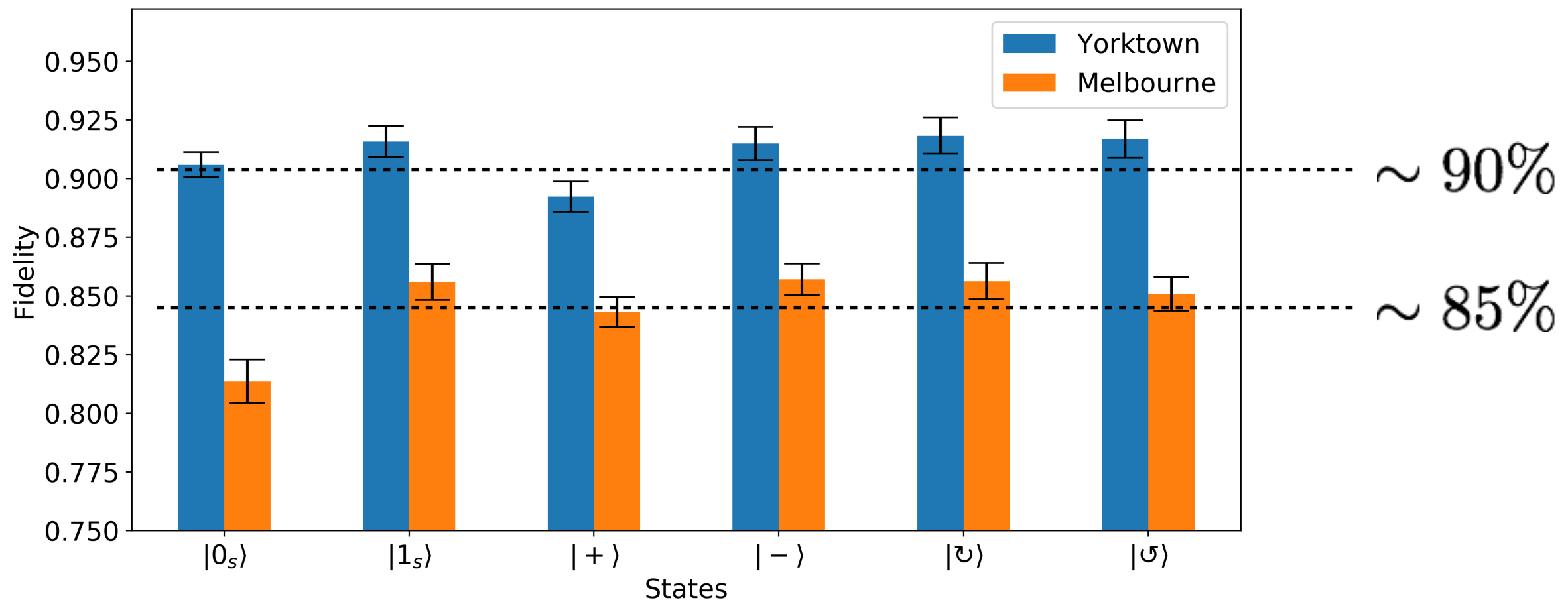
Fidelities

Classical fidelity:

$$F(p, q) = \sum_i \sqrt{p_i q_i}$$

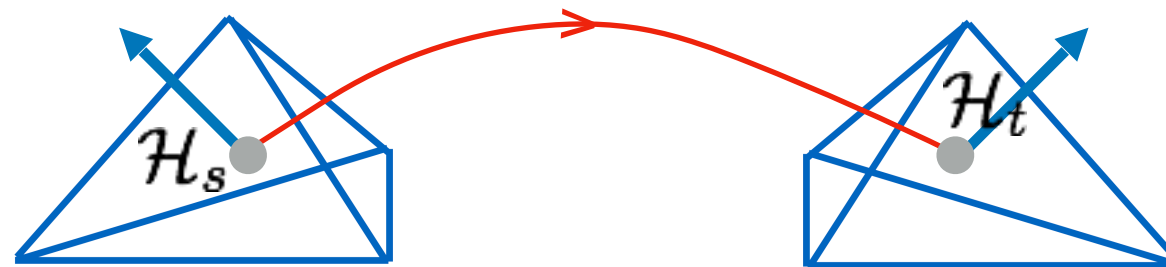
Experimental results:

State	Yorktown	Melbourne
$ 0_s\rangle$	0.906 ± 0.005	0.814 ± 0.009
$ 1_s\rangle$	0.916 ± 0.007	0.856 ± 0.008
$ +\rangle$	0.892 ± 0.007	0.843 ± 0.006
$ -\rangle$	0.915 ± 0.007	0.857 ± 0.007
$ \odot\rangle$	0.918 ± 0.008	0.856 ± 0.008
$ \oslash\rangle$	0.917 ± 0.008	0.851 ± 0.007



Beyond a single node...

SU(2) holonomies = maximal entanglement



Quantum entanglement is „gluing” together faces of tetrahedra.

The state associated with holonomy can be written as:

$$|\mathcal{E}\rangle = \frac{1}{\sqrt{2}} h_{IJ}^* |I\rangle_s |J\rangle_t \in \mathcal{H}_s \otimes \mathcal{H}_t$$

$$\text{e.g. } |\mathcal{E}_l\rangle = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle)$$

h_{IJ} are matrix components of the SU(2) holonomy.

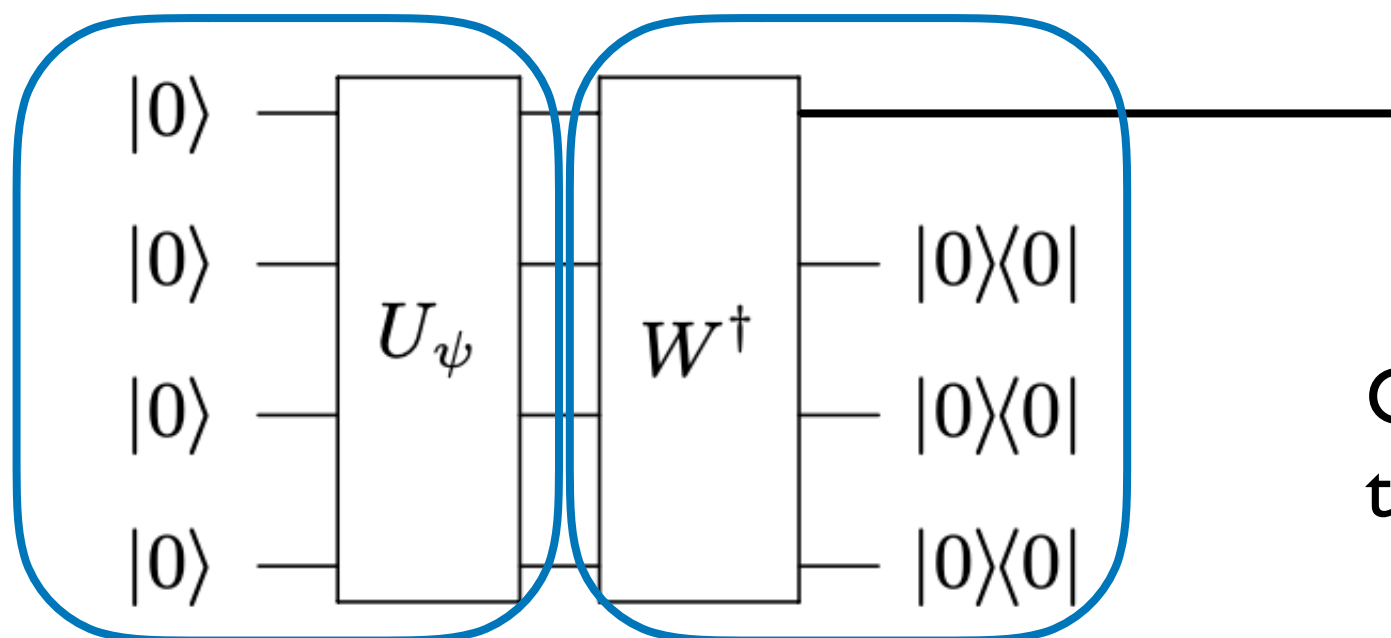
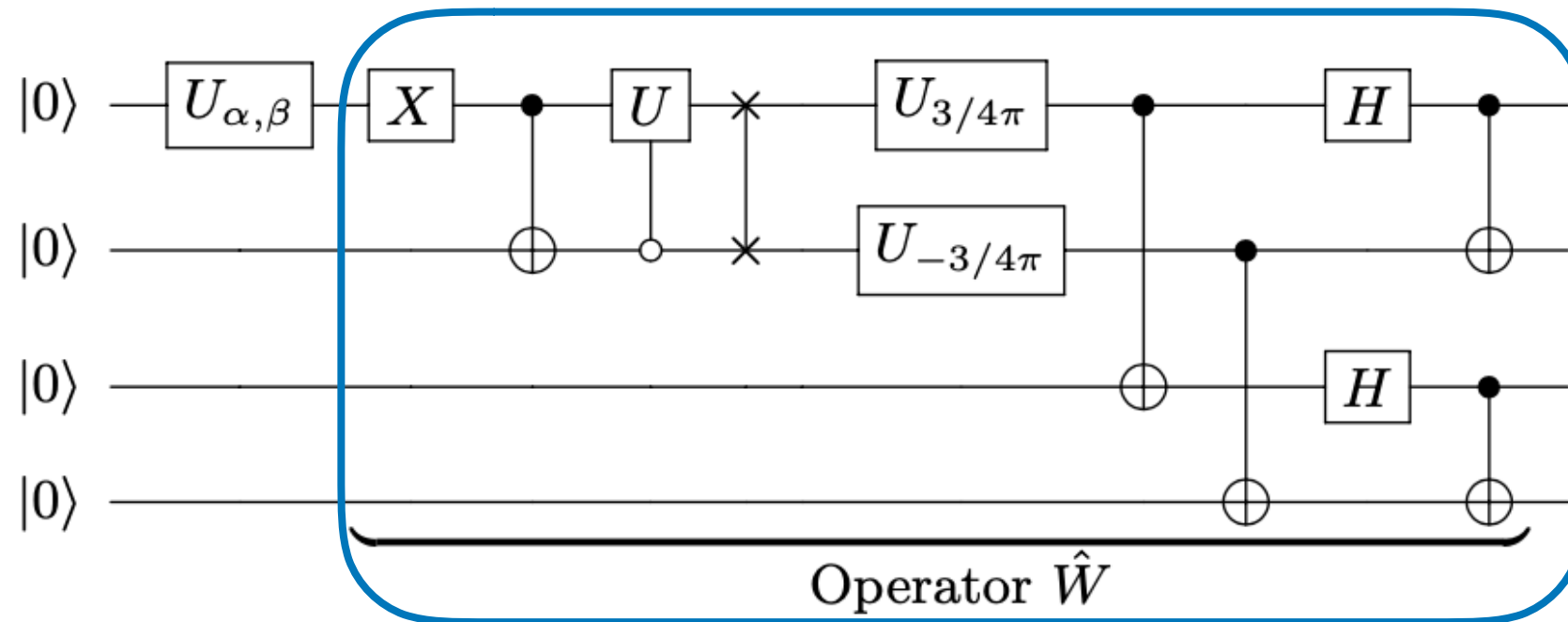
Based on this, **Maximally Entangled Spin Network (MESN)** states can be introduced:

$$|\text{MESN}\rangle := \hat{P}_G \bigotimes_l |\mathcal{E}_l\rangle$$

New circuit for an Ising node

G. Czelusta & JM (2023)

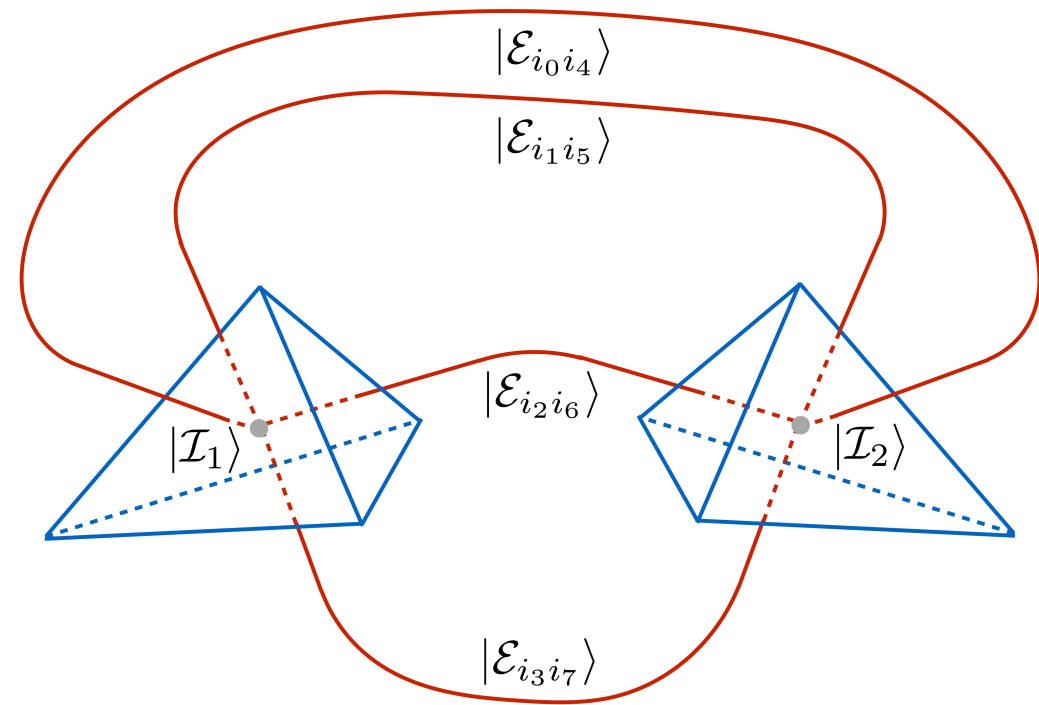
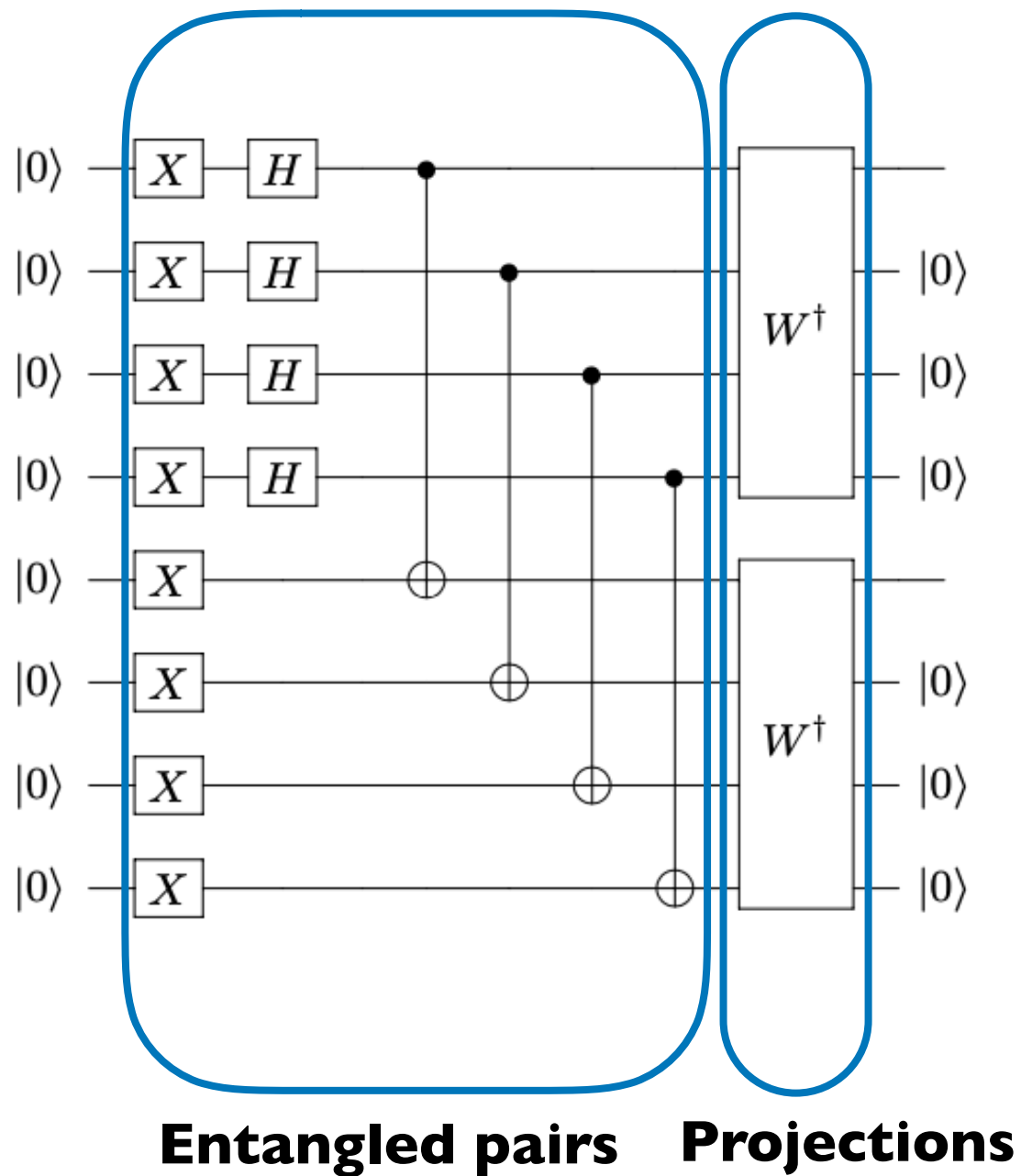
$$\hat{W} (\alpha|0\rangle + \beta|1\rangle) |000\rangle = |\mathcal{I}(\alpha, \beta)\rangle = \alpha|\iota_0\rangle + \beta|\iota_1\rangle$$



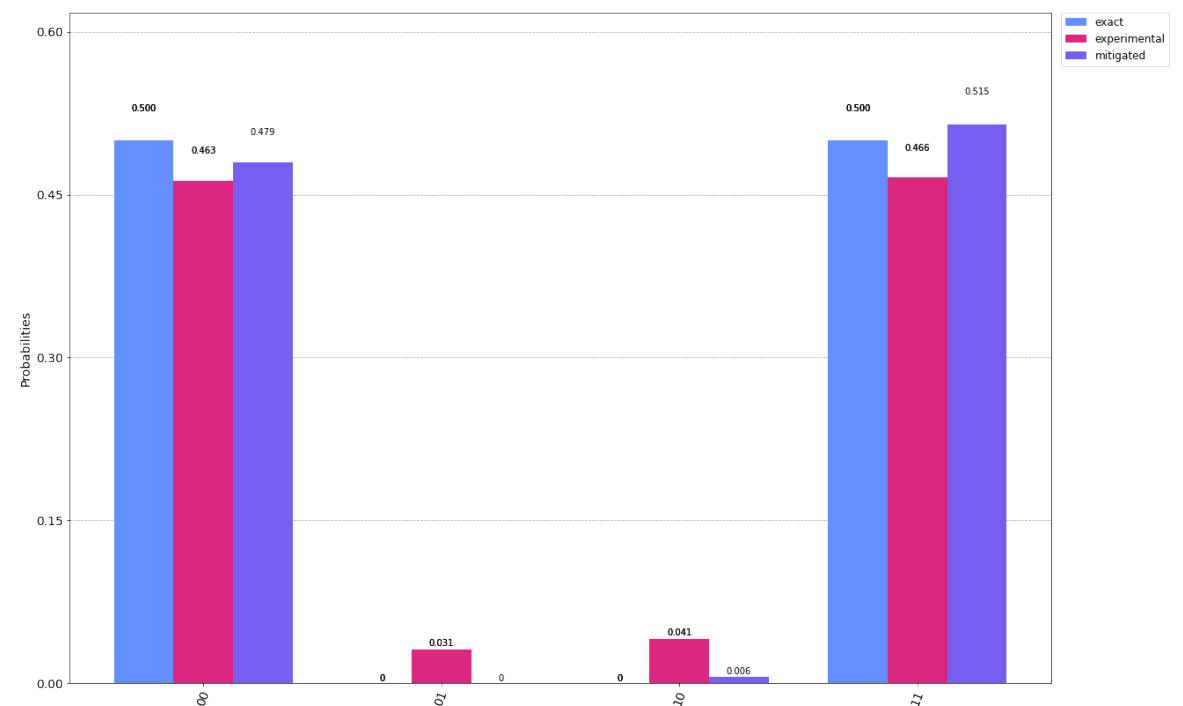
Operator $\hat{W}$ contributes to a „projection” operator

State preparation Projection

Dipole



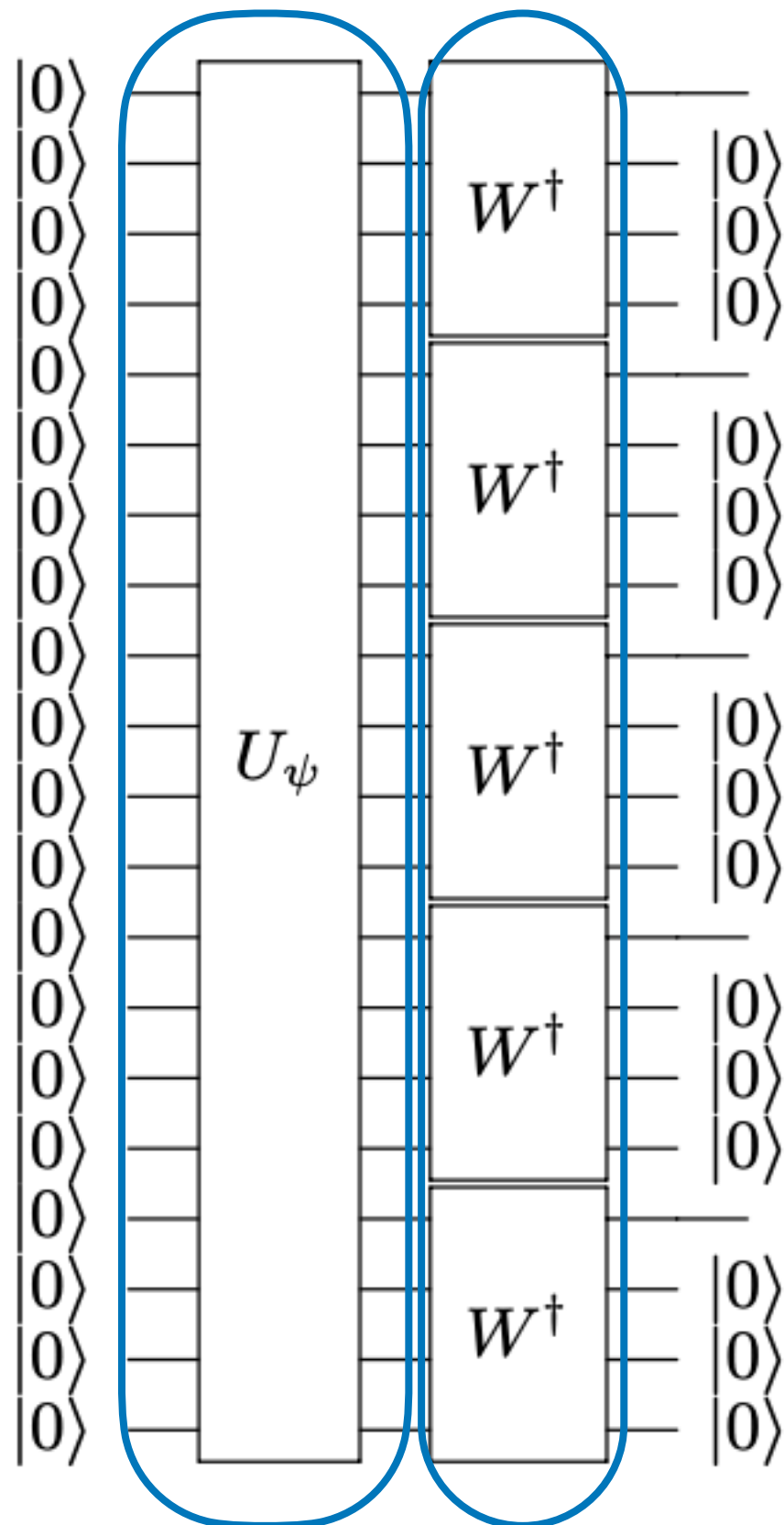
Measured and predicted probabilities:



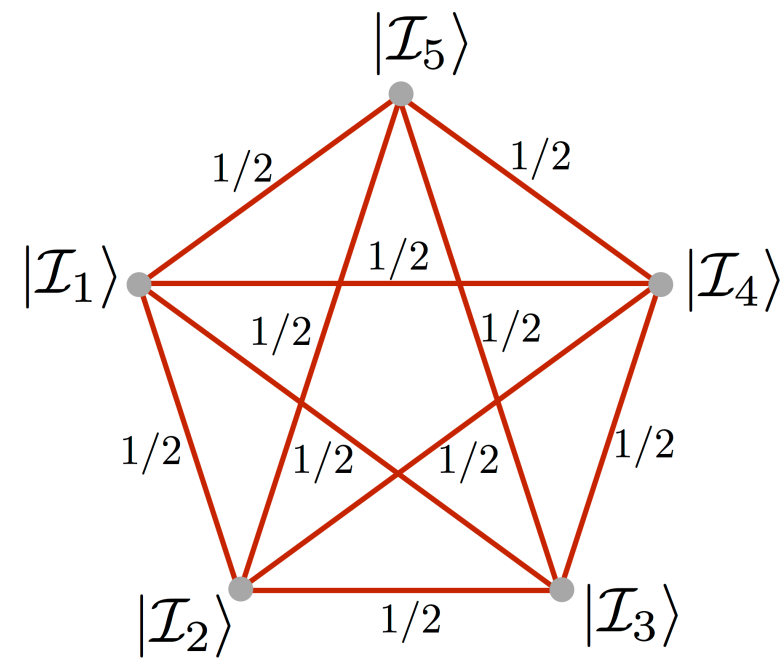
Manila IBM quantum computer

The quantum fidelity of the found state is ≈ 0.99

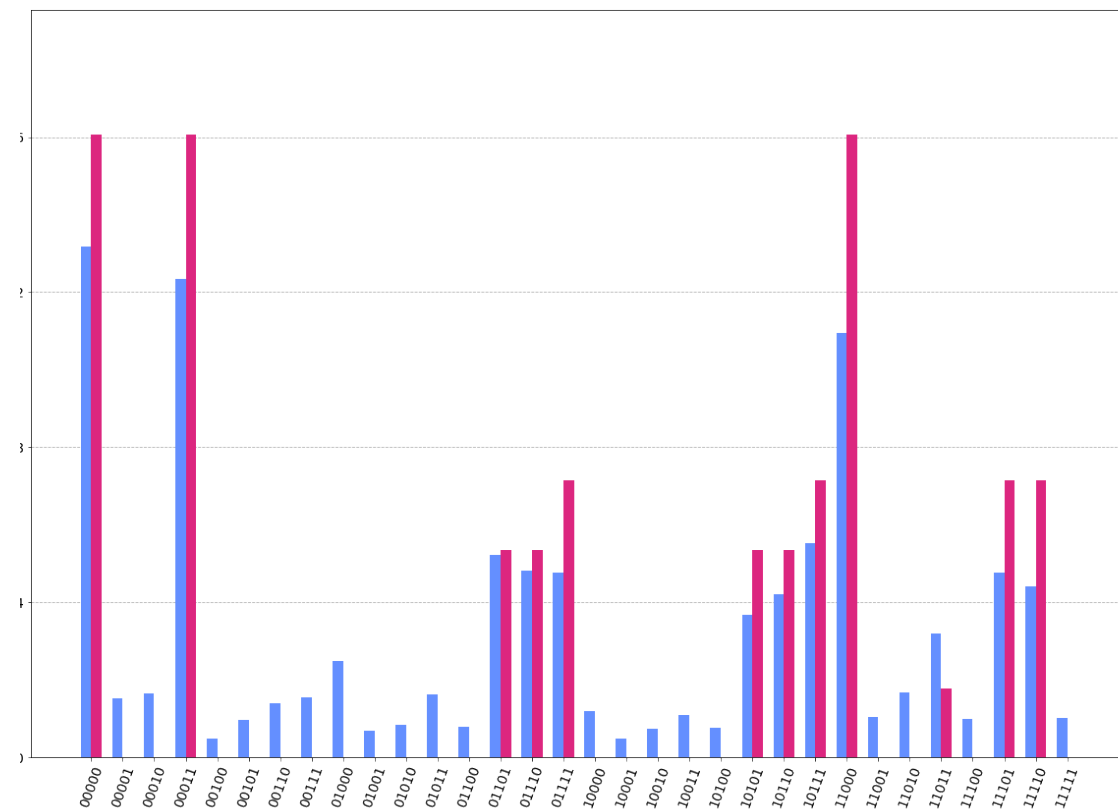
Pentagram



Entangled pairs Projections



Measured and predicted (from the $\{15j\}$ symbol) probabilities:

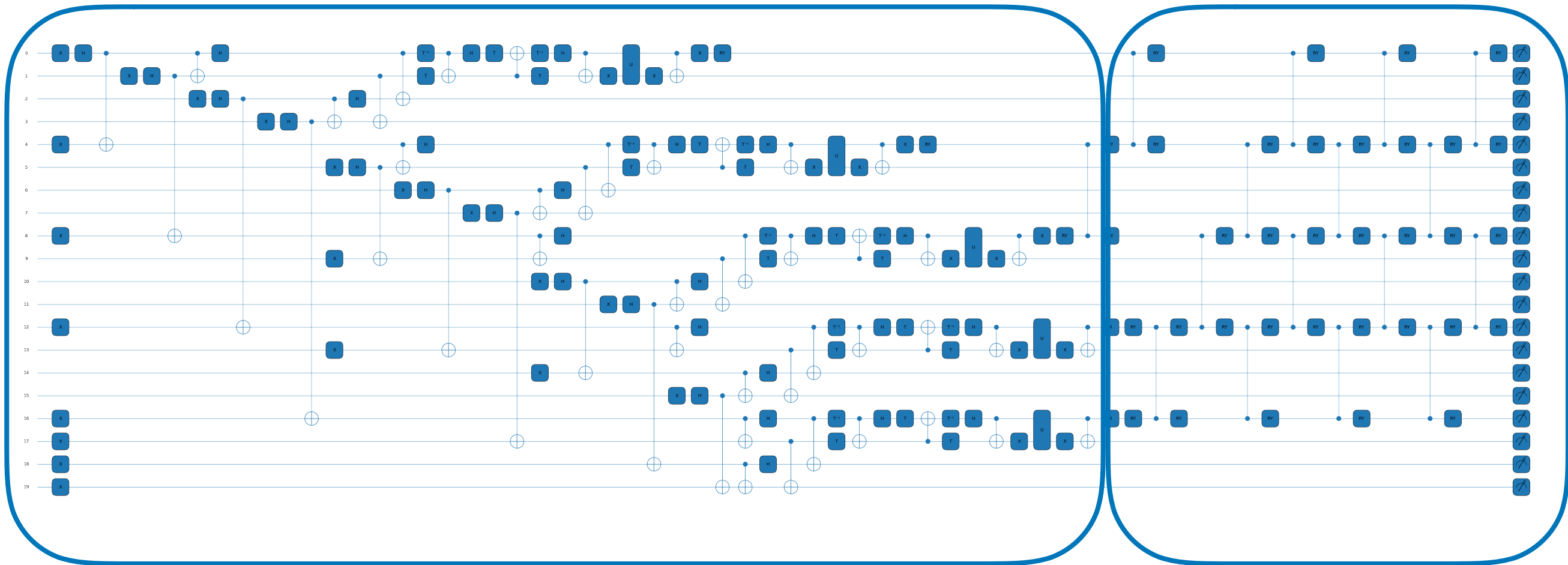


Manila IBM quantum computer

The quantum fidelity of the found state is: ≈ 0.77

Variational transfer of the 5-qubit state of the pentagram

Ansatz



Pentagram on 20 quits

Pentagram on 5 quits

The probability of the state $|0\rangle^{\otimes 20}$ is maximized.

Employing the transfer method, **arbitrary 4-valent (Ising) spin network states can be effectively constructed** [G. Czelusta & JM (2023)]

Furthermore, using recoupling theory of the $SU(2)$ group, **spin networks can be expressed in terms of products and superpositions of the Wilson loops** and in consequence as the function of $|\Psi\rangle$.

Namely, since $|\Psi\rangle = \frac{1}{\sqrt{2}} h_{IJ}^* |I\rangle_s |J\rangle_t \in \mathcal{H}_s \otimes \mathcal{H}_t$

we find $h_{IJ} = \sqrt{2} (({}_s\langle I| {}_t\langle J|) |\Psi\rangle)^* = \sqrt{2} \langle \Psi | (|I\rangle_s |J\rangle_t)$

In consequence, the Wilson loop is:

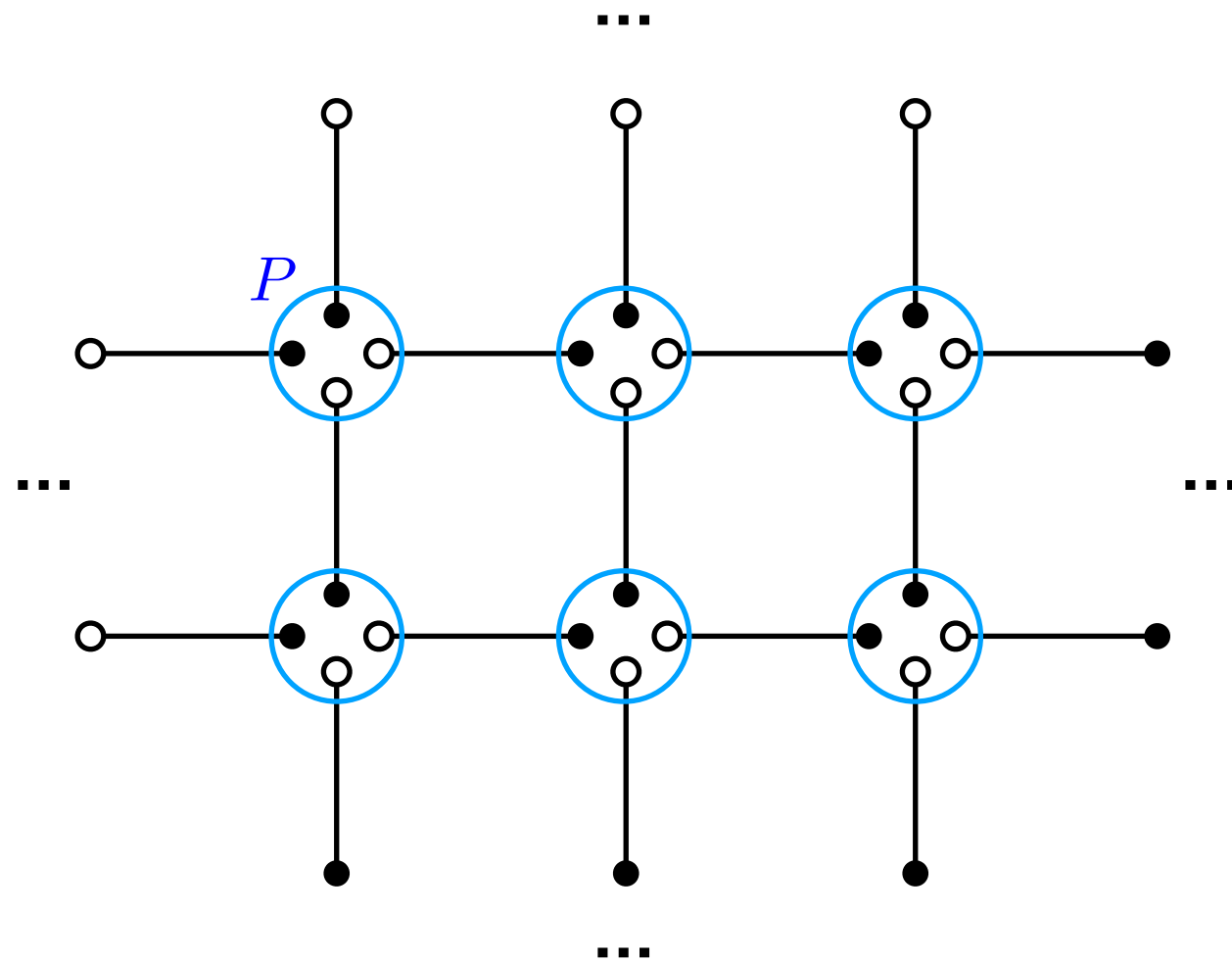
$$W_e[h] = \sum_I h_{II} = \sqrt{2} \sum_I \langle \Psi | (|I\rangle |I\rangle)$$

[JM & T.Trześniewski (2020)]

Relation to tensor networks

Tensor network

(PEPS - Projected Entangled Pair States)

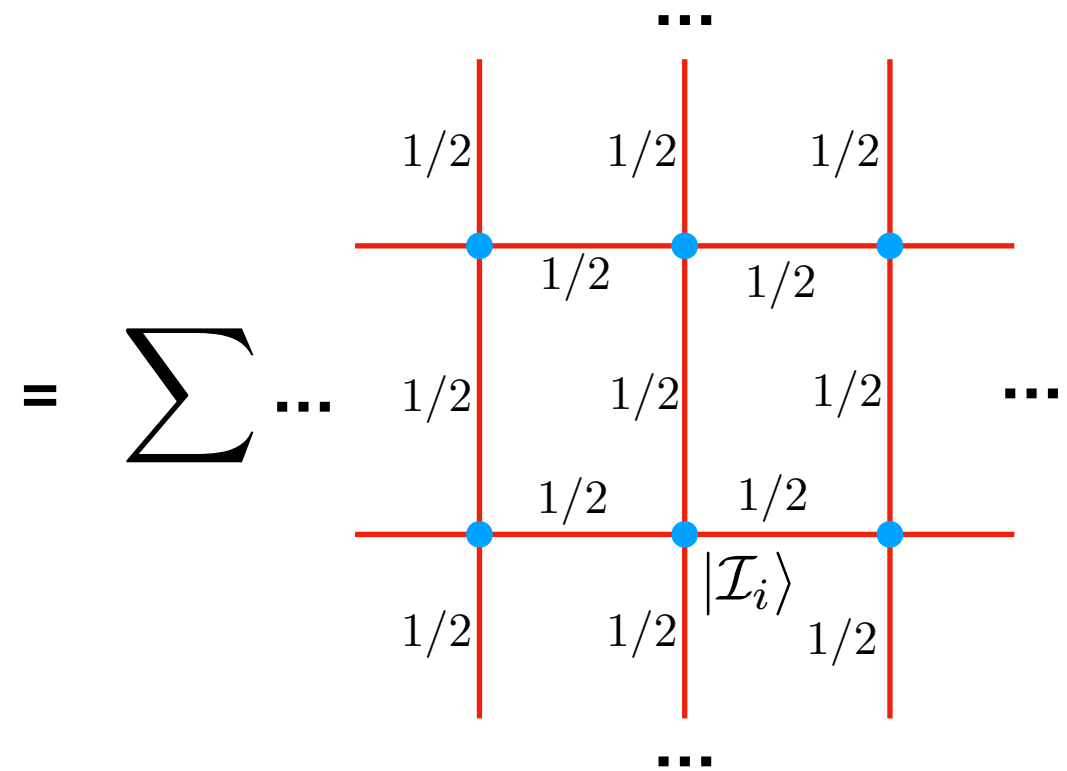


$$\text{white circle} \text{---} \text{black dot} = |\mathcal{E}_l\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

P - projection onto spin-0 subspace of four spins (dim = 2)

Spin network

(Ising type MESN with $j=1/2$)



$|\mathcal{I}_i\rangle$ - intertwiner qubit

$$|\Psi\rangle = \sum_{1,2,\dots,n} a_{12\dots n} |\mathcal{I}_1\rangle |\mathcal{I}_2\rangle \dots |\mathcal{I}_n\rangle$$

Summary and future prospects

- Holonomies of $SU(2)$ gauge field carry maximal entanglement.
- Gauge invariant states of the discrete $SU(2)$ gauge theory can be introduced and represented as quantum circuits.
- First quantum simulations of $SU(2)$ gauge invariant states have successfully been performed on quantum computers.
- The quantum computing methods may bring advantage to simulations of the gauge theories - computational complexity to be explored (e.g. using geometric methods).
- Implementation of quantum dynamics is to be done.
- Extension of the construction to other gauge fields, e.g. $SU(3)$ and beyond (large N limit) is an exciting research challenge.

Thank you!



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