

Classical Entanglement and Entropy

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Entanglement and high energy collisions?

Kharzeev and Levin, 2017 piqued our interest in possible effects of quantum entanglement in high energy collisions

DIS probes a small area of the proton, the degrees of freedom in this area are entangled with the rest of the proton. The entropy of this entanglement is manifested as the entropy of particle production in the collision?

KL: especially interesting in the saturation regime: the hadron state is "maximally entangled".

Not the only and not the first idea involving entanglement in hadron collisions.

Eigenstate thermalization: collision produces a system in a highly excited state, in states like these a large set of observable is expected to have distributions close to thermal. Again-subset of degrees of freedom. The entropy associated with this quasi thermal behavior of the subset is actually entanglement entropy with the rest of the degrees of freedom.

Is this so very quantum?

Saturation regime - **Classical** Color Glass Condensate description

Eigenstate Thermalization - highly excited states, quite possibly in the WKB regime, i.e. semiclassical.

Is this physics really quantum?

Can we understand at least part of the "entanglement" in question classically?

The simplest exercise: take two coupled oscillators that move along a **single** classical trajectory. The two oscillators are "classically entangled", i.e. the probability to measure one oscillator at a given point depends on the coordinates of the other oscillator. Form the probability distribution for the coordinate and momentum of one of the oscillators, and calculate its Boltzmann entropy.

Compare this "classical entanglement" entropy to the quantum von Neumann entropy of the same quantum system in a highly excited state.

Single oscillator: the probability distribution

Single oscillator first:

$$H(x, p) = \frac{p^2}{2} + V(x)$$

Energy conservation:

$$\mathcal{W}(p, x) = F(x, p) \delta[H(x, p) - E]$$

Determine function F : $\mathcal{W}(x)$ is proportional to the time the particle spends in the interval Δx around the point x as it traverses the trajectory many times.

$$\mathcal{W}(x) \equiv \int dp \mathcal{W}(x, p); \propto \Delta t = \frac{\Delta x}{|\dot{x}|} = \frac{\Delta x}{|p(x)|}$$

Similarly

$$\mathcal{W}(p) \propto \frac{\Delta p}{|\dot{p}|} = \frac{\Delta p}{\left| \frac{\partial H}{\partial x}(x(p), p) \right|}$$

Compare and normalize:

$$\mathcal{W}(x, p) = \frac{\omega}{\pi} \delta \left[\frac{p^2}{2} + \frac{\omega^2}{2} x^2 - E \right].$$

Classical oscillators: the probability distribution

Coupled classical oscillators

$$\mathcal{H}_c = \frac{1}{2} [p_x^2 + p_y^2 + \omega^2 x^2 + \Omega^2 y^2 + 2Cxy] .$$

We take $\Omega \gg \omega \gg C$ and integrate over the "fast" coordinate y .

What is the joint probability distribution? One could diagonalize the system and write the probability distribution as the product for the two independent oscillators. Alternatively, there are two conserved quantities in the system :

$$E_+ = \frac{1}{2} [p_x^2 + p_y^2 + \omega^2 x^2 + \Omega^2 y^2 + 2Cxy] ,$$
$$E_- \approx \frac{1}{2} [p_y^2 - p_x^2] + \frac{2C}{\Omega^2} p_x p_y + \frac{1}{2} [\Omega^2 y^2 - \omega^2 x^2] + Cxy .$$

\mathcal{W} must be proportional to the product of δ -functions. Determine the proportionality constant from normalization. So:

$$\mathcal{W}(x, p_x; y, p_y) = N \delta \left(E_+ - \frac{1}{2} [p_x^2 + p_y^2 + \omega^2 x^2 + \Omega^2 y^2 + 2Cxy] \right)$$
$$\times \delta \left(E_- - \left[\frac{1}{2} [p_y^2 - p_x^2] + \frac{2C}{\Omega^2} p_x p_y + \frac{1}{2} [\Omega^2 y^2 - \omega^2 x^2] + Cxy \right] \right)$$

The "reduced" probability density.

Now for the "reduced" probability distribution. I.e. if we only measure the coordinate x and momentum p_x , we have to "integrate out" the coordinate y , p_y .

$$\mathcal{W}(x, p_x) = N \int_{y, p_y} \delta \left(E_+ - \frac{1}{2} [p_x^2 + p_y^2 + \omega^2 x^2 + \Omega^2 y^2 + 2Cxy] \right) \\ \times \delta \left(E_- - \left[\frac{1}{2} [p_y^2 - p_x^2] + \frac{2C}{\Omega^2} p_x p_y + \frac{1}{2} [\Omega^2 y^2 - \omega^2 x^2] + Cxy \right] \right).$$

Luckily for us when $C \ll \omega^2 \ll \Omega^2$ integrating out y , p_y is easy analytically

$$\mathcal{W}(x, p_x) = N \frac{\Omega^2}{2C|p_x|} \frac{2}{\sqrt{C^2 x^2 + \Omega^2 (2E_+ - (p_x^2 + \bar{p}_y^2 + \omega^2 x^2))}}$$

where

$$\bar{p}_y = \frac{\Omega^2}{2Cp_x} (E_- - E_+ + p_x^2 + \omega^2 x^2).$$

The Boltzmann Entropy

"Classical entanglement" entropy (the Boltzmann entropy for our "reduced" probability distribution)

$$S_B = - \int dp dx \mathcal{W}(p, x) \ln[\mathcal{W}(p, x)\Delta].$$

The parameter Δ needs to be introduced since \mathcal{W} is dimensionless! The usual "complication" with classical continuous distributions.

This is actually nontrivial, and this is what distinguishes Boltzmann entropy from Shannon entropy in discrete systems.

We take $\Delta = h/2$ having in mind quantum-classical correspondence. More on this later if time permits.

The Boltzmann entropy

The integral again can be performed analytically. Long story short:

$$S_{CE} = \ln \left[\frac{\pi^2 C}{\Omega^2} \frac{\sqrt{E_+^2 - E_-^2}}{h\omega} \right] = \ln \left[\frac{\pi C}{\Omega} \frac{\sqrt{E_1 E_2}}{\hbar\omega\Omega} \right].$$

here E_1 and E_2 are energies of the two noninteracting "normal mode" oscillators into which we can rotate x and y .

For C too small this is negative. It happens with Boltzmann entropy: e.g. "third law" of thermodynamics is violated in classical statistical mechanics.

The condition of the positivity of entropy in fact coincides with the requirement that the interaction energy of the two oscillators is much larger than $\hbar\Omega$.

The quantum system

Take the same quantum system.

Take a highly excited state, where occupation numbers for both oscillators are large.

Integrate out the y oscillator.

Calculate von Neumann entropy

If we are to relate the classical entropy to the entropy of the quantum system, we should require that all energies are large relative to the vacuum energy of a single oscillator:

$$E_1 \gg \hbar\Omega, \quad E_2 \gg \hbar\Omega; \quad C^2 \langle x^2 y^2 \rangle \sim C^2 \frac{E_1 E_2}{\Omega^2 \omega^2} \gg \hbar^2 \Omega^2 .$$

High energy -highly excited states - WKB wave functions.

WKB 101 - Harmonic oscillator

For a single oscillator

$$\phi_{n, \text{ odd}}^{\text{WKB}}(x) = \sqrt{\frac{4}{T\rho(x)}} \sin\left(\frac{S(x)}{\hbar}\right), \quad \phi_{n, \text{ even}}^{\text{WKB}}(x) = \sqrt{\frac{4}{T\rho(x)}} \cos\left(\frac{S(x)}{\hbar}\right).$$

$$\rho(x) \equiv \sqrt{2 \left(E_n - \frac{1}{2} \omega^2 x^2 \right)}.$$

$$S(x) = \int_0^x dz p(z) = \sqrt{2} \int_0^x dz \sqrt{E_n - \frac{1}{2} \omega^2 z^2}$$

The action can be written

$$S(x) = n\left(\theta + \frac{1}{2} \sin(2\theta)\right); \quad \sin(\theta) = \sqrt{\frac{\hbar\omega}{2n}} x.$$

For x far away from the classical turning points $|x|_{\text{turn}} = \sqrt{\frac{2n+1}{\hbar\omega}}$,

$$S(x) \approx x\sqrt{2n\hbar\omega}.$$

This is actually a very good approximation for almost all x .

WKB- two coupled oscillators

We start with

$$\phi_{n,m}^{\text{WKB}}(x,y) = \sqrt{\frac{16}{T_1 T_2 p(x_1) p(x_2)}} \sin\left(\frac{1}{\hbar} S_1(x_1)\right) \sin\left(\frac{1}{\hbar} S_2(x_2)\right)$$

The "normal modes":

$$x_1 = \alpha x - \beta y, \quad x_2 = \beta x + \alpha y$$

with

$$\alpha^2 \approx 1 - \frac{C^2}{\Omega^4}, \quad \beta^2 \approx \frac{C^2}{\Omega^4}.$$

For $C \ll \omega^2 \ll \Omega^2$

$$\begin{aligned} \phi_{n,m}^{\text{WKB}}(x,y) &\approx \sqrt{\frac{16}{T_1 T_2 p_x p_y}} \sin\left(\frac{1}{\hbar} S_1(x)\right) \sin\left(\frac{1}{\hbar} (S_2(y) + \beta x p_y)\right) \\ &= \sqrt{\frac{16}{T_1 T_2 p_x p_y}} \sin\left(\frac{1}{\hbar} S_1(x)\right) \left[\sin\frac{1}{\hbar} S_2(y) \cos\frac{1}{\hbar} \beta x p_y + \cos\frac{1}{\hbar} S_2(y) \sin\frac{1}{\hbar} \beta x p_y \right] \end{aligned}$$

The reduced density matrix I.

By definition:

$$\hat{\rho}(x, \bar{x}) = \int dy \phi_{n,m}^{\text{WKB}*}(\bar{x}, y) \phi_{n,m}^{\text{WKB}}(x, y).$$

y has high frequency, oscillates fast and all oscillating terms can be neglected, except those that come with prefactor β in the exponent:

$$\begin{aligned} \hat{\rho}(x, \bar{x})^{\text{WKB}} \approx & \int dy \frac{2}{T_1 T_2 p_y} \frac{1}{\sqrt{p_x p_{\bar{x}}}} \\ & \left\{ \left[\sin \left(\frac{1}{\hbar} [S_1(x) + \beta x p_y] \right) + \sin \left(\frac{1}{\hbar} [S_1(x) - \beta x p_y] \right) \right] \right. \\ & \times \left[\sin \left(\frac{1}{\hbar} [S_1(\bar{x}) + \beta \bar{x} p_y] \right) + \sin \left(\frac{1}{\hbar} [S_1(\bar{x}) - \beta \bar{x} p_y] \right) \right] \\ & + \left[\cos \left(\frac{1}{\hbar} [S_1(x) + \beta x p_y] \right) - \cos \left(\frac{1}{\hbar} [S_1(x) - \beta x p_y] \right) \right] \\ & \left. \times \left[\cos \left(\frac{1}{\hbar} [S_1(\bar{x}) + \beta \bar{x} p_y] \right) - \cos \left(\frac{1}{\hbar} [S_1(\bar{x}) - \beta \bar{x} p_y] \right) \right] \right\}. \end{aligned} \quad (1)$$

The reduced density matrix II

Define

$$\delta n = \frac{1}{\hbar} \sqrt{\frac{2n}{\omega}} \beta p_y.$$

Parametrically $p_y \sim \sqrt{2E_2}$, so that

$$\delta n \sim \frac{C}{\Omega} \frac{\sqrt{E_1 E_2}}{\Omega \hbar \omega}; \quad \delta n \ll n.$$

We can then write

$$\begin{aligned} \hat{\rho}^{\text{WKB}}(x, \bar{x}) &= \int dy \frac{2}{T_1 T_2 p_y} \frac{1}{\sqrt{p_x p_{\bar{x}}}} \\ &\left\{ \left[\sin \frac{1}{\hbar} x \sqrt{2(n + \delta n)\omega} + \sin \frac{1}{\hbar} x \sqrt{2(n - \delta n)\omega} \right] \right. \\ &\times \left[\sin \frac{1}{\hbar} \bar{x} \sqrt{2(n + \delta n)\omega} + \sin \frac{1}{\hbar} \bar{x} \sqrt{2(n - \delta n)\omega} \right] \\ &+ \left[\cos \frac{1}{\hbar} x \sqrt{2(n + \delta n)\omega} - \cos \frac{1}{\hbar} x \sqrt{2(n - \delta n)\omega} \right] \\ &\times \left. \left[\cos \frac{1}{\hbar} \bar{x} \sqrt{2(n + \delta n)\omega} - \cos \frac{1}{\hbar} \bar{x} \sqrt{2(n - \delta n)\omega} \right] \right\}. \end{aligned}$$

Looks almost like weighted integral over the occupation number of products of WKB functions for x .

Reduced density matrix, finally

With a little bit more massaging

$$\hat{\rho}^{\text{WKB}}(x, \bar{x}) = \int_0^{\sqrt{\frac{4E_1 E_2 C^2}{\hbar^2 \omega^2 \Omega^4}}} d(\delta n) \lambda(\delta n) \sum_i \phi_{\delta n}^{i*}(\bar{x}) \phi_{\delta n}^i(x)$$

with the basis functions

$$\begin{aligned}\phi_{\delta n}^1 &\equiv \sqrt{\frac{1}{T_1 \rho(x)}} \left[\sin\left(\frac{1}{\hbar} S_{n+\delta n}(x)\right) + \sin\left(\frac{1}{\hbar} S_{n-\delta n}(x)\right) \right] \\ \phi_{\delta n}^2 &\equiv \sqrt{\frac{1}{T_1 \rho(x)}} \left[\cos\left(\frac{1}{\hbar} S_{n+\delta n}(x)\right) - \cos\left(\frac{1}{\hbar} S_{n-\delta n}(x)\right) \right]\end{aligned}$$

and the normalized probability density

$$\lambda(\delta n) = \sqrt{\frac{\hbar^2 \omega^2 \Omega^4}{4\pi^2 C^2 E_1 E_2}} \frac{1}{\sqrt{1 - \frac{\hbar^2 \omega^2 \Omega^4}{4E_1 E_2 C^2} (\delta n)^2}}.$$

The quantum entanglement entropy.

Once we have the density matrix as the weighted sum of products of basis functions, the entropy is easy

$$\begin{aligned} S_E &= -2 \int_0^{\sqrt{\frac{4E_1 E_2 C^2}{\hbar^2 \omega^2 \Omega^4}}} d(\delta n) \lambda(\delta n) \ln \lambda(\delta n) \\ &= \ln \left[\frac{\pi C}{\Omega} \frac{\sqrt{E_1 E_2}}{\hbar \omega \Omega} \right]. \end{aligned}$$

Compare with our result for the "classical entanglement" entropy

$$S_{CE} = \ln \left[\frac{\pi C}{\Omega} \frac{\sqrt{E_1 E_2}}{\hbar \omega \Omega} \right] = S_E$$

Discussion/Conclusions

An observation. For the ground state the entanglement entropy has been calculated earlier. Easy for excited states with low energies E_1 and E_2 , such that

$$f = \frac{C^2 E_1 E_2}{(\hbar\omega)^2 \Omega^4} \ll 1$$

In this regime reduced density matrix mixes only two states, and

$$S_E^{low} = -(1-f) \ln(1-f) - f \ln f$$

f is the "occupation number" of the x oscillator.

Compare this with our result ($f > 1$)

$$S_E^{high} = \ln(\pi f^{1/2}).$$

We are in the "maximally entangled" regime - many states ($\delta n \sim f^{1/2}$) are occupied with (almost) equal probability.

Discussion/Conclusion

What does it teach us?

There exists quantum-classical correspondence for entanglement entropy, at least in our simple case.

The analog of the "maximally entangled state" in our model is indeed classical, and can be studied classically.

For this system classical entanglement captures the essence of quantum entanglement.

Disclaimer: of course not every quantum entangled state has a classical analog. Bell's theorem remains Bell's theorem.

Is the same true in QCD, i.e. for eigenstate thermalization and such? I wish I knew, but it is not out of the question.

After conclusions: Shannon vs Boltzmann entropy

Entropy for **smooth** distribution of a continuous variable is tricky.

$$S^a = - \int d^D V \mathcal{W}(x_i) \ln[\mathcal{W}(x_i) \Delta V]$$

with $\Delta V = a^D$. No "continuum limit" $a \rightarrow 0$, i.e. Shannon entropy diverges for a smooth distribution.

If L - a typical scale of variation of \mathcal{W} ,

$$S^a = S(L) - D \ln \frac{a}{L}$$

With $S(L)$ - finite and well defined.

After conclusions: Shannon vs Boltzmann entropy

Is our classical oscillator a "pure state"? No, in the sense that it has nonvanishing Shannon or Boltzmann entropy.

\mathcal{W} is not smooth - recall a δ -function. But if we use for a the same scale on which we regulate the δ -function we get

$$S_{SO} = -\ln \left[\frac{a}{\pi} \sqrt{\frac{2\omega}{E}} \right] \quad S_{DO} = -\ln \left[\frac{a^2}{\pi^2} \sqrt{\frac{4\omega_1\omega_2}{E_1 E_2}} \right].$$

This is nonzero, but "unnaturally small" in the sense that it scales with the cutoff as $\frac{D}{2} \ln a$ rather than $D \ln a$.

After conclusions: Shannon vs Boltzmann entropy

Can classical entropy grow when we integrate degrees of freedom? I.e. can we characterize "classical entanglement" by entropy growth like in QM?

If yes, it cannot be Shannon entropy. For a distribution of discrete variable.

$$S^{Shannon} = - \sum_x p_x \ln p_x$$

If $W(x) \equiv \sum_y W(x, y)$ it is always true:

$$S_2^{Shannon} = \sum_{x,y} W(x, y) \ln W(x, y) \geq S_1^{Shannon} = \sum_x W(x) \ln W(x).$$

But for Boltzmann entropy

$$S_1^{Boltzmann} - S_2^{Boltzmann} = S_1^{Shannon} - S_2^{Shannon} + \ln \frac{b}{a}$$

Since $b > a$, there is no sharp statement.

In fact, if we define **entropy per degree of freedom**, for our example of two harmonic oscillators it indeed grows when we integrate out y , precisely because a single oscillator is "almost a pure state".