Entanglement Entropy in Particle Scattering

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This talk is based on the works with R. Peschanski (IPhT, CEA-Saclay):

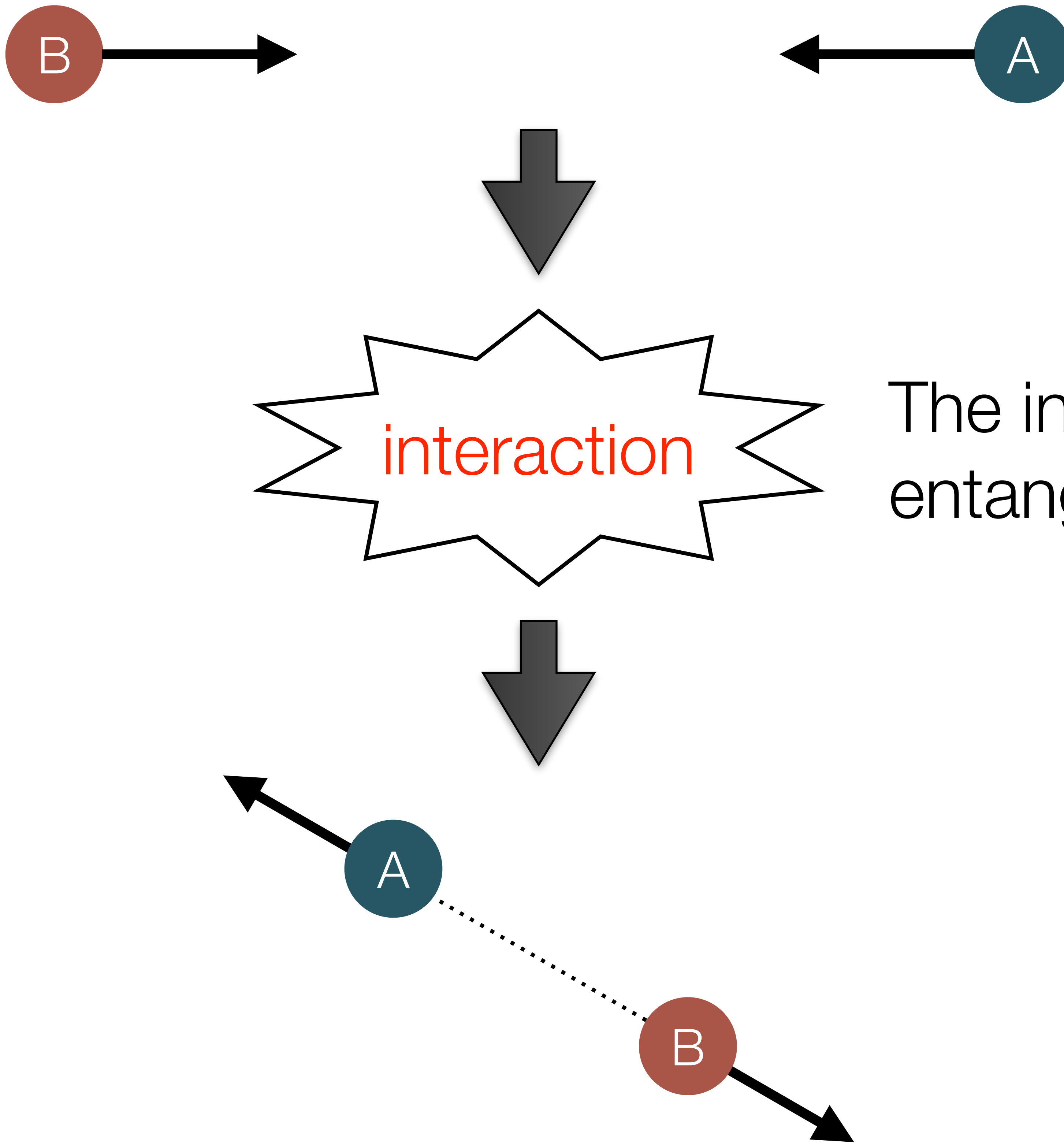
"Entanglement Entropy of Scattering Particles"
Phys. Lett. B758 (2016) 89 [arXiv:1602.00720],

"Evaluation of Entanglement Entropy in High Energy Elastic Scattering"
Phys. Rev. D 100 (2019) 076012 [arXiv:1906.09696].

Quantum Entanglement in High Energy Physics 2023 10 May 2023

Question: What is the entanglement entropy of the final state in elastic scattering?

an unentangled pair of incoming particles



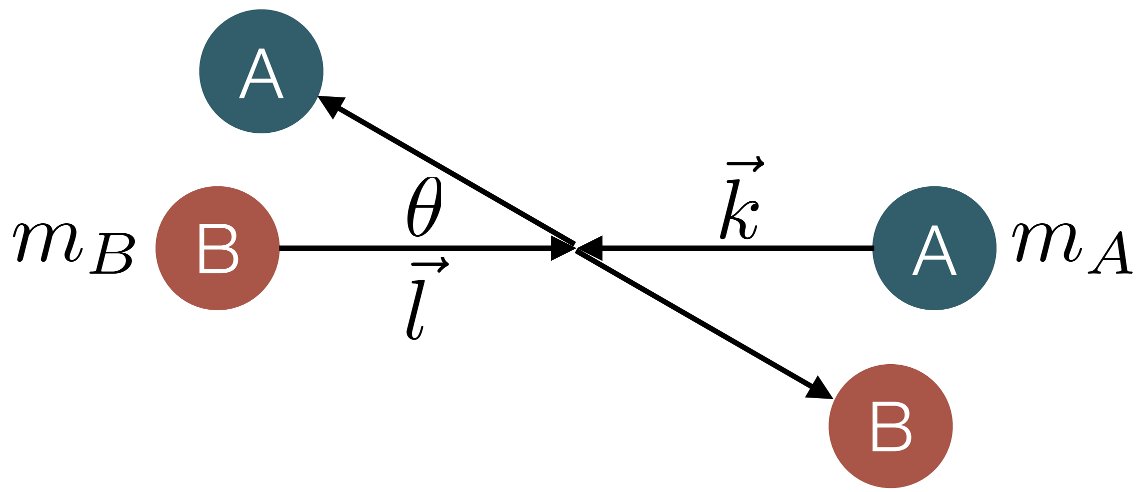
The interaction produces entanglement.

an entangled pair of outgoing particles

Formulation of entanglement entropy in elastic scattering

[Peschanski and Seki, Phys. Lett. B758 (2016) 89]

Let us consider an elastic scattering of two particles, A and B.



We focus on a two-particle state in the momentum Hilbert space.

[Seki and Sin, Phys. Lett. B735 (2014) 272]

[Balasubramanian, McDermott and Raamsdonk, Phys. Rev D86 (2012) 045014]

$$|\vec{p},\vec{q}\rangle \equiv |\vec{p}\rangle_A \otimes |\vec{q}\rangle_B \in \mathcal{H}_A \otimes \mathcal{H}_B$$

$$\langle \vec{p},\vec{q}|\vec{k},\vec{l}\rangle = 2E_{A\vec{p}}\,\delta^{(3)}(\vec{p}-\vec{k})\,2E_{B\vec{q}}\,\delta^{(3)}(\vec{q}-\vec{l})$$

$$E_{I\vec{p}} = \sqrt{p^2+m_I^2} \quad (I=A,B)$$

Initial state $|\text{ini}\rangle = |\vec{k}, \vec{l}\rangle$

S-matrix $\mathcal{S}$

Once we fix the initial state, the S-matrix, $\mathcal{S}$, gives the final state; $\mathcal{S}|\text{ini}\rangle$. However it includes not only two-particle states.

Two-particle final state $|\text{fin}\rangle$

Since we are interested in the final state of two particles, we project out the states except for the two-particle ones by using the projection operator;

$$|\text{fin}\rangle = \int \frac{d^3\vec{p}}{2E_{A\vec{p}}} \frac{d^3\vec{q}}{2E_{B\vec{q}}} |\vec{p}, \vec{q}\rangle\langle\vec{p}, \vec{q}|\mathcal{S}|\vec{k}, \vec{l}\rangle$$

Total density matrix ρ

The total density matrix for the final state is defined by

$$\rho \equiv \frac{1}{\mathcal{N}} |\mathrm{fin}\rangle\langle \mathrm{fin}|$$

$$= \frac{1}{\mathcal{N}} \int \frac{d^3\vec{p}}{2E_{A\vec{p}}} \frac{d^3\vec{q}}{2E_{B\vec{q}}} \frac{d^3\vec{p}\,'}{2E_{A\vec{p}\,'}} \frac{d^3\vec{q}\,'}{2E_{B\vec{q}\,'}} |\vec{p},\vec{q}\rangle\langle\vec{p},\vec{q}|\mathcal{S}|\vec{k},\vec{l}\rangle\langle\vec{k},\vec{l}|\mathcal{S}^\dagger|\vec{p}\,',\vec{q}\,'\rangle\langle\vec{p}\,',\vec{q}\,'|$$

$\mathcal{N}$ is a normalization factor. Later it will be determined by $\mathrm{tr}_A\,\mathrm{tr}_B\,\rho = 1$.

Reduced density matrix ρ_A

Tracing out the total density matrix with respect to $\mathcal{H}_B$, we obtain the reduced density matrix;

$$\rho_A \equiv \mathrm{tr}_B\,\rho = \int \frac{d^3\vec{q}\,''}{2E_{B\vec{q}\,''}}\, {}_B\langle\vec{q}\,''|\rho|\vec{q}\,''\rangle_B$$

$$= \frac{1}{\mathcal{N}} \int \frac{d^3\vec{p}}{2E_{A\vec{p}}} \frac{d^3\vec{q}}{2E_{B\vec{q}}} \frac{d^3\vec{p}\,'}{2E_{A\vec{p}\,'}} \left(\langle\vec{p},\vec{q}|\mathcal{S}|\vec{k},\vec{l}\rangle\langle\vec{k},\vec{l}|\mathcal{S}^\dagger|\vec{p}\,',\vec{q}\rangle\right)|\vec{p}\rangle_{A\,A}\langle\vec{p}\,'|$$

We extract a factor about the energy-momentum conservation from the S-matrix element,

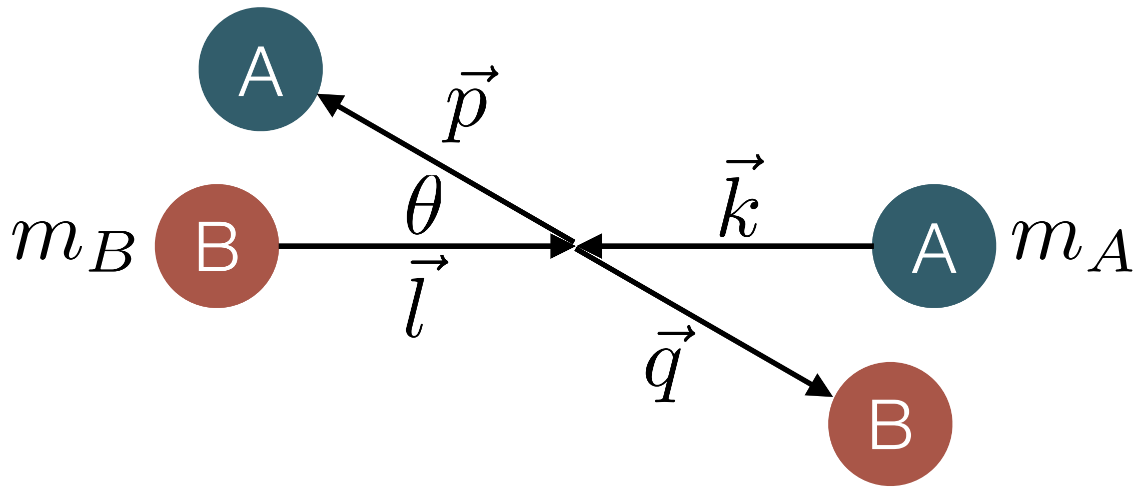
$$\langle \vec{p}, \vec{q} | \mathcal{S} | \vec{k}, \vec{l} \rangle = \delta^{(4)}(P^{(4)}_{p+q} - P^{(4)}_{k+l}) \, \langle \vec{p}, \vec{q} | \mathbf{s} | \vec{k}, \vec{l} \rangle \, ,$$

where $P^{(4)}$ stands for the center-of-mass energy momentum vector.

Then we can rewrite the reduced density matrix as

$$\rho_A = \frac{1}{\mathcal{N}} \int \frac{d^3\vec{p}}{2E_{A\vec{p}}} \frac{\delta(0)\delta(E_{A\vec{p}} + E_{B\vec{k}+\vec{l}-\vec{p}} - E_{A\vec{k}} - E_{B\vec{k}})}{2E_{A\vec{p}} 2E_{B\vec{k}+\vec{l}-\vec{p}}}$$
$$\times \Big(\langle \vec{p}, \vec{k} + \vec{l} - \vec{p} | \mathbf{s} | \vec{k}, \vec{l} \rangle \langle \vec{k}, \vec{l} | \mathbf{s}^\dagger | \vec{p}, \vec{k} + \vec{l} - \vec{p} \rangle \Big) | \vec{p} \rangle_{A\,A} \langle \vec{p} |$$

Now let us move into the center-of-mass frame.



$$\vec{k}+\vec{l}=\vec{p}+\vec{q}=0$$

$$p:=|\vec{p}|\,,\qquad k:=|\vec{k}|\,,$$

$$\frac{\vec{p}\cdot\vec{k}}{pk}=\cos\theta$$

The normalization factor determined by $\mathrm{tr}_A\,\mathrm{tr}_B\,\rho=1$ is

$$\mathcal{N}=\delta^{(4)}(0)\,\mathcal{N}'\,,\qquad \mathcal{N}'=\int d^3\vec{p}\;\frac{\delta(p-k)}{4k(E_{A\vec{k}}+E_{B\vec{k}})}\left|\langle\vec{p},-\vec{p}|\mathbf{s}|\vec{k},-\vec{k}\rangle\right|^2$$

The reduced density matrix is rewritten as

$$\rho_A=\frac{1}{\mathcal{N}'\delta^{(3)}(0)}\int\frac{d^3\vec{p}}{2E_{A\vec{p}}}\;\frac{\delta(p-k)}{4k(E_{A\vec{k}}+E_{B\vec{k}})}\left|\langle\vec{p},-\vec{p}|\mathbf{s}|\vec{k},-\vec{k}\rangle\right|^2|\vec{p}\rangle_A\,{}_A\langle\vec{p}|$$

In order for the entanglement entropy, we calculate

$$\mathrm{tr}_A(\rho_A)^n = \int d^3\vec{p}\,\delta^{(3)}(0)\left(\delta(p-k)\frac{\left|\langle\vec{p},-\vec{p}\,|s|\vec{k},-\vec{k}\rangle\right|^2}{\mathcal{N}'\delta^{(3)}(0)4k(E_{A\vec{k}}+E_{B\vec{k}})}\right)^n$$

because

$$S_{\mathrm{EE}} = -\lim_{n\to 1}\frac{\partial}{\partial n}\,\mathrm{tr}_A(\rho_A)^n = -\,\mathrm{tr}_A\,\rho_A\ln\rho_A$$

(*cf*. Rényi entropy: $S_{\mathrm{RE}} = \dfrac{1}{1-n}\ln\mathrm{tr}_A(\rho_A)^n$)

Partial wave expansion

[Van Hove, Nuovo Cim. 28 (1963) 2344]

[Białas and Van Hove, Nuovo Cim. 38 (1965) 1385]

The partial wave expansion is often useful for analyzing scattering processes.

$$\langle \vec{p}, -\vec{p} | \mathbf{s} | \vec{k}, -\vec{k} \rangle = \frac{E_{A\vec{k}} + E_{B\vec{k}}}{\pi k} \cdot \sum_{\ell=0}^{\infty} (2\ell + 1)(1 + 2i\tau_\ell) P_\ell(\cos\theta)$$

$$= \frac{E_{A\vec{k}} + E_{B\vec{k}}}{\pi k} \cdot 2\left(\delta(1 - \cos\theta) + \frac{i}{16\pi}\mathcal{A}(s,t)\right)$$

$$s_\ell = 1 + 2i\tau_\ell$$

P_ℓ are Legendre polynomials.

We know the summation formula of Legendre polynomial,

$$\delta(1 - \cos\theta) = \frac{1}{2}\sum_{\ell=0}^{\infty}(2\ell + 1)P_\ell(\cos\theta)$$

One can calculate $\mathrm{tr}_A(\rho_A)^n$ in terms of the partial wave modes,

$$\mathrm{tr}_A(\rho_A)^n = \left(\frac{\delta(0)}{\delta^{(3)}(0)2\pi k^2}\right)^{n-1} \int_{-1}^{1} d\cos\theta \left[\frac{1}{2}\frac{\left|\sum_\ell (2\ell+1)s_\ell P_\ell(\cos\theta)\right|^2}{\sum_\ell (2\ell+1)|s_\ell|^2}\right]^n$$

By using the mathematical identity of delta-function in spherical coordinates with azimuthal symmetry,

$$\delta^{(3)}(\vec{p}-\vec{k}) = \frac{\delta(p-k)}{4\pi k^2} \sum_{\ell=0}^{\infty}(2\ell+1)P_\ell(\cos\theta)$$

we obtain

$$2\pi k^2 \, \frac{\delta^{(3)}(0)}{\delta(0)} = \frac{1}{2}\sum_{\ell=0}^{\infty}(2\ell+1) = \frac{V}{2}$$

V is the divergent full phase-space "volume":

$$V \equiv \sum_{\ell=0}^{\infty}(2\ell+1) = 2\delta(0)$$

We obtain the following expression:

$$\mathrm{tr}_A(\rho_A)^n = \left(\frac{V}{2}\right)^{1-n} \int_{-1}^{1} d\cos\theta\, [\mathcal{P}(\theta)]^n\,,$$

$$\mathcal{P}(\theta) = \frac{1}{2}\frac{\left|\sum_\ell (2\ell+1) s_\ell P_\ell(\cos\theta)\right|^2}{\sum_\ell (2\ell+1)|s_\ell|^2}$$

One can recognize it as a kind of probability. Actually it is of norm,

$$\int_{-1}^{1} d\cos\theta\, \mathcal{P}(\theta) = 1$$

$\mathcal{P}(\theta)$ is also written in terms of τ_ℓ, f_ℓ as

$$\mathcal{P}(\theta) = \delta(1-\cos\theta)\left(1 - \frac{2\sum_\ell (2\ell+1)|\tau_\ell|^2}{V/2 - \sum_\ell (2\ell+1) f_\ell}\right) + \frac{\left|\sum_\ell (2\ell+1)\tau_\ell P_\ell(\cos\theta)\right|^2}{V/2 - \sum_\ell (2\ell+1) f_\ell}$$

f_ℓ $(\equiv (\mathrm{Im}\,\tau_\ell - |\tau_\ell|^2)$ from the unitarity condition of the S-matrix) corresponds to inelastic channels.

The cross sections in terms of the partial wave modes are

$$\sigma_{\mathrm{el}} = \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)|\tau_\ell|^2\,, \quad \sigma_{\mathrm{inel}} = \frac{2\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1) f_\ell\,,$$

$$\sigma_{\mathrm{tot}} = \sigma_{\mathrm{el}} + \sigma_{\mathrm{inel}} = \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\,\mathrm{Im}\,\tau_\ell\,,$$

$$\frac{d\sigma_{\mathrm{el}}}{dt} = \frac{\pi}{k^4}\left|\sum_\ell (2\ell+1)\tau_\ell P_\ell(\cos\theta)\right|^2 = \frac{|\mathcal{A}|^2}{256\pi k^4}$$

(t is the Mandelstam variable; $t = 2k^2(\cos\theta - 1)$.)

One can rewrite $\mathcal{P}(\theta)$ as

$$\mathcal{P}(\theta) = \delta(1-\cos\theta)\cdot\left(1-\frac{\sigma_{\rm el}}{\pi V/k^2-\sigma_{\rm inel}}\right) + \frac{1}{\sigma_{\rm el}}\frac{d\sigma_{\rm el}}{d\cos\theta}\cdot\left(\frac{\sigma_{\rm el}}{\pi V/k^2-\sigma_{\rm inel}}\right)$$

Entanglement entropy

$$S_{\rm EE} = -\lim_{n\to 1}\frac{\partial}{\partial n}\,{\rm tr}_A(\rho_A)^n = \ln\frac{V}{2} - \int_{-1}^{1} d\cos\theta\,\mathcal{P}(\theta)\ln\mathcal{P}(\theta)$$

There is a problem, *i.e.*, this formula depends on the infinite volume V, so that this is physically meaningless. Therefore some regularization is necessary.

Regularization

Volume regularization

$$\mathcal{P}(\theta) = \delta(1-\cos\theta)\cdot\left(1-\frac{\sigma_{\rm el}}{\pi V/k^2-\sigma_{\rm inel}}\right) + \frac{1}{\sigma_{\rm el}}\frac{d\sigma_{\rm el}}{d\cos\theta}\cdot\left(\frac{\sigma_{\rm el}}{\pi V/k^2-\sigma_{\rm inel}}\right)$$

The first term comes from part of the two-body Hilbert space of the final states which does not correspond to the interacting states at the given energy. Because this term has support only at $\theta = 0$.

cf.

$$\langle\vec{p}, -\vec{p}|\mathbf{s}|\vec{k}, -\vec{k}\rangle = \frac{E_{A\vec{k}}+E_{B\vec{k}}}{\pi k}\cdot 2\left(\delta(1-\cos\theta)+\frac{i}{16\pi}\mathcal{A}(s,t)\right)$$

In order to avoid the non-interacting modes in an ideal cut-off independent way, we regularize the volume V to $\tilde{V}$,

$$1 - \frac{\sigma_{\rm el}}{\pi \tilde{V}/k^2 - \sigma_{\rm inel}} = 0 \Leftrightarrow \tilde{V} = \frac{k^2(\sigma_{\rm el} + \sigma_{\rm inel})}{\pi} = \frac{k^2 \sigma_{\rm tot}}{\pi}$$

and then

$$\tilde{\mathcal{P}}(\theta) = \frac{1}{\sigma_{\rm el}} \frac{d\sigma_{\rm el}}{d\cos\theta} = \frac{2k^2}{\sigma_{\rm el}} \frac{d\sigma_{\rm el}}{dt}$$

Currently we do not know a concrete way to realize this regularization. So we call it “ideal”.

Finally this regularization leads the formal entanglement entropy to the volume-regularized entanglement entropy,

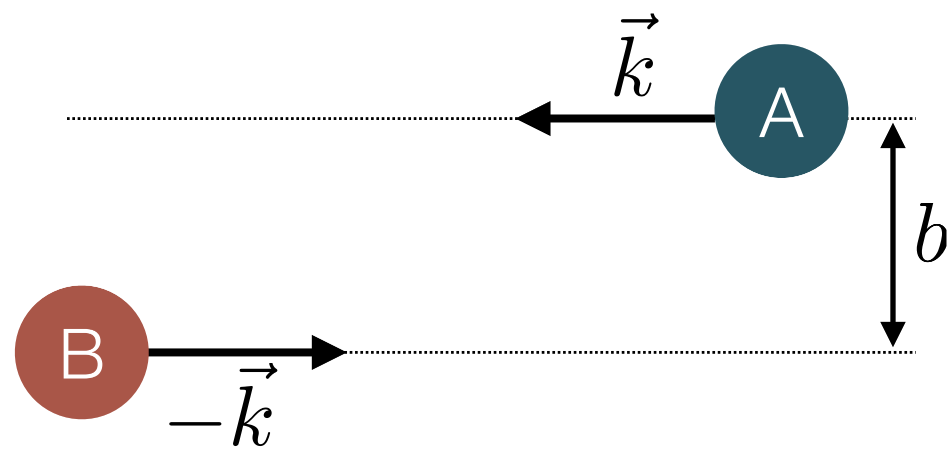
$$\tilde{S}_{\rm EE} = -\int_{-\infty}^{0} dt \, \frac{1}{\sigma_{\rm el}} \frac{d\sigma_{\rm el}}{dt} \ln\left(\frac{4\pi}{\sigma_{\rm tot}\sigma_{\rm el}} \frac{d\sigma_{\rm el}}{dt} \right)$$

Cutoff regularization

Impact parameter representation

$$\mathcal{A} = 16\pi \sum_{\ell=0}^{\infty} (2\ell+1)\tau_\ell P_\ell(\cos\theta) = 32\pi k^2 \int_0^\infty b db\, \tau(b) J_0(b\sqrt{-t})$$

Bessel function



In order to get rid of the contribution of large b, we introduce a cutoff function $c(b)$ with $\lim_{b\to 0} c(b) = 0$.

$$\hat{\mathcal{A}} = 32\pi k^2 \int_0^\infty b db\, c(b) \tau(b) J_0(b\sqrt{-t})$$

For example,

- step-function

$$c(b) = \begin{cases} 1 & (b \leq 2\Lambda) \\ 0 & (b > 2\Lambda) \end{cases}$$

- Gaussian

$$c(b) = \exp\left(-\frac{1}{2} \cdot \frac{b^2}{4\Lambda^2} \right)$$

The cross sections are also modified as

$$\hat{\sigma}_{\rm tot} = 8\pi \int_0^\infty bdb\, c^2(b)\, {\rm Im}\, \tau(b)\,,$$

$$\hat{\sigma}_{\rm el} = \int_{-\infty}^0 dt\, \frac{d\hat{\sigma}_{\rm el}}{dt} = 8\pi \int_0^\infty bdb\, c^2(b) |\tau(b)|^2\,,$$

$$\hat{\sigma}_{\rm inel} = 4\pi \int_0^\infty bdb\, c^2(b) f(b)\,,$$

$$\frac{d\hat{\sigma}_{\rm el}}{dt} = 4\pi \left| \int_0^\infty bdb\, c(b) \tau(b) J_0(b\sqrt{-t}) \right|^2$$

For both cutoff functions, the infinite volume is regularized as

$$V \to \hat{V} = 2k^2 \int_0^\infty bdb\, c^2(b) = 4k^2\Lambda^2$$

and also we impose the condition;

$$\mathcal{P}(\theta) = \delta(1-\cos\theta)\cdot\left(1 - \frac{\sigma_{\rm el}}{\pi\hat{V}/k^2 - \sigma_{\rm inel}}\right) + \frac{1}{\sigma_{\rm el}}\frac{d\sigma_{\rm el}}{d\cos\theta}\cdot\left(\frac{\sigma_{\rm el}}{\pi\hat{V}/k^2 - \sigma_{\rm inel}}\right)$$

$$= 0$$

$$\Rightarrow\ \hat{V} = \frac{k^2}{\pi}\hat{\sigma}_{\rm tot}$$

Hence the cutoff parameter Λ is fixed,

$$4\pi\Lambda^2 = \hat{\sigma}_{\rm tot}$$

Evaluation of Entanglement Entropy in Proton-Proton Scattering at High Energy

[Peschanski and Seki, Phys. Rev. D 100 (2019) 076012]

Let us evaluate the regularized entanglement entropy for high-energy proton-proton scattering.
One can use the experimental data of the elastic, inelastic and total cross sections by Tevatron and LHC.

In order to use our formula with the volume regularization, we need to know the differential cross section $\frac{d\sigma_{\rm el}}{dt}$ as a function of t.

Therefore, for simplicity, we assume a diffraction peak approximation, which is characterized by the scattering amplitude:

$$\mathcal{A}(s,t) = is\sigma_{\rm tot}e^{\frac{1}{2}Bt}$$

B is a slope parameter.

s is the Mandelstam variable equal to (center-of-mass energy)^2.

The differential cross section and the slope parameter are

$$\frac{d\sigma_{\rm el}}{dt} = \frac{\sigma_{\rm tot}^2}{16\pi} e^{Bt} , \qquad B = \frac{\sigma_{\rm tot}^2}{16\pi\sigma_{\rm el}}$$

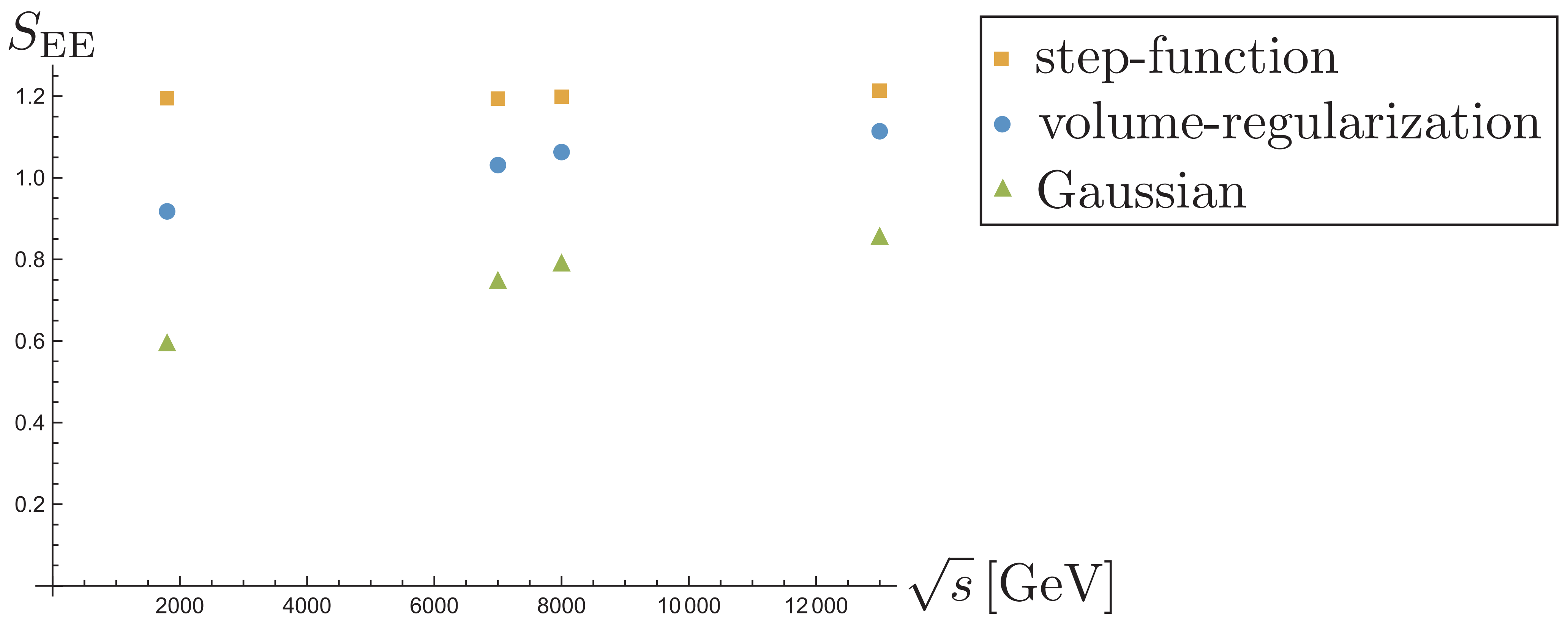
We can calculate the volume-regularized entanglement entropy,

$$\tilde{S}_{\rm EE} = -\int_{-\infty}^{0} dt \, \frac{1}{\sigma_{\rm el}} \frac{d\sigma_{\rm el}}{dt} \ln\left(\frac{4\pi}{\sigma_{\rm tot}\sigma_{\rm el}} \frac{d\sigma_{\rm el}}{dt} \right) = 1 + \ln \frac{4\sigma_{\rm el}}{\sigma_{\rm tot}}$$

$\sqrt{s}$ [GeV]	$\sigma_{\rm tot}$ [mb]	$\sigma_{\rm el}$ [mb]	$\tilde{S}_{\rm EE}$
1800	72.1	16.6	0.9176
7000	98.58	25.43	1.031
8000	101.7	27.1	1.063
13000	110.6	31.0	1.114

The entropy monotonically increases as the center-of-mass energy.

Comparison of the entanglement entropy for proton-proton scattering in 3 regularization schemes.



In all cases, the entanglement entropy monotonically increases as the center-of-mass energy.
The difference among the 3 regularization schemes shrinks at higher energy.

Summary and future problems

Summary

- We have formulated the entanglement entropy of the two-particle final state in an elastic scattering.
- The divergence of infinite volume is regularized in the "ideal" cutoff-independent way.
- Assuming the diffraction peak model, we have evaluated the regularized entanglement entropy for the high-energy proton-proton scattering by using the experimental data by Tevatron and LHC.
- The entanglement entropy monotonically increases as the center-of-mass energy.

Future problems

- Inelastic scattering [work in progress with R. Peschanski]

$$
\begin{aligned}
A_1 + B_1 \to & A_1 + B_1 \qquad \text{(elastic)} \\
& A_2 + B_1 \qquad \text{(two-particle inelastic)} \\
& X \qquad \text{(multi-particle inelastic)}
\end{aligned}
$$

entanglement entropies and their ratio?

$$
S_{A_1+B_1}\,, \;\; S_{A_2+B_2}\,, \qquad \frac{S_{A_1+B_1}}{S_{A_2+B_2}}
$$

- Diffraction dissociation
- AdS/CFT correspondence
- String scattering