Complexity in spin chain

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Based on : N.Jaiswal, M.Gautam, T.Sarkar [Jaiswal et al., 2022]

Introduction/Motivation

- Susskind [Brown et al., 2016] introduced holographic complexity as the boundary entity growth corresponds to the evolution of the Einstein-Rosen bridge
- It's an efficient probe of novel phenomena such as quantum chaos and quantum phase transition
- In computer science, the notion of complexity refers to the minimum number of operations required to implement a task.
- In terms of the quantum circuit model, we have to construct a unitary transformation which produces a target state by acting on a simple reference state.

$$|\psi_T\rangle = U\,|\psi_R\rangle \tag{1}$$

- Nielsen [Nielsen, 2005] geometrize this idea by finding the cost function in unitary space, minimizing the cost function gives geodesic connecting the two states.

Quantum Information Tools

We defined some of the information tools :

- Quantum Geometric Tensor: QGT introduced by Provost and Vallee [Provost and Vallee, 1980], where they defined it as the distance between two states in projective Hilbert space.

$$\chi_{ij} = \langle \partial_i \psi | \partial_j \psi \rangle - \langle \partial_i \psi | \psi \rangle \langle \psi | \partial_j \psi \rangle$$

  $g_{ij} = Re[\chi_{ij}]$

- Loschmidt Echo: It quantifies the deviation of the time-evolved state from the initial condition.

$$\mathcal{G}(t) = \langle \psi_0 | \psi_0(t) \rangle = \langle \psi_0 | e^{-iHt} | \psi_0 \rangle$$

$$\mathcal{L}(t) = |\mathcal{G}(t)|^2$$

Model

- Hamiltonian for XY spin chain

$$H = -\sum_{l=-M}^{l=M} \left(\frac{1+\gamma}{4}\sigma_l^x \sigma_{l+1}^x + \frac{1-\gamma}{4}\sigma_l^y \sigma_{l+1}^y - \frac{h}{2}\sigma_l^z\right)$$

- Energy Spectrum:

$$\epsilon_k = \pm\sqrt{(\cos k + h)^2 + (\gamma \sin k)^2}$$

- Ground state:

$$|\psi_o\rangle_{h,\gamma} = \prod_{k>0} \left[\cos\left(\frac{\theta_k}{2}\right)|0,0\rangle - i\sin\left(\frac{\theta_k}{2}\right)|1,-1\rangle\right]$$

where

$$\cos\theta_k = \frac{\cos k + h}{\sqrt{(\cos k + h)^2 + (\gamma \sin k)^2}}$$

Nielsen Complexity

- Nielsen Complexity defined as:

$$\mathcal{C}_N = \sum_k |\Delta\theta_k|^2$$

where

$$\Delta\theta_k = \frac{\theta_k^T - \theta_k^R}{2}$$

- θ_k^T : corresponds to target state , such that $\theta(h^T, \gamma^T)$
- θ_k^R : corresponds to reference state , such that $\theta(h^R, \gamma^R)$

- Target state parameter : $h^T = h + \delta, \quad \gamma^T = \gamma$
- Reference state parameter : $h^R = h, \quad \gamma^R = \gamma$
- Complexity

$$\mathcal{C}_N\big|_{|h|<1} = \frac{\delta^2}{16|\gamma|(1-h^2)} + \frac{h\delta^3}{16|\gamma|(1-h^2)^2} + O(\delta^4)$$

$$\mathcal{C}_N\big|_{|h|>1} = \frac{|h|\delta^2\gamma^2}{16(h^2-1)(h^2+\gamma^2-1)^{3/2}} + O(\delta^3)$$

Component of metric tensor [Kolodrubetz et al., 2013];

$$g_{hh}\big|_{|h|<1} = \frac{1}{16|\gamma|(1-h^2)} \quad g_{hh}\big|_{|h|>1} = \frac{|h|\gamma^2}{16(h^2-1)(h^2-1+\gamma^2)^{3/2}}$$

Quantum quench

- Initial ground state $|\psi_0\rangle$ of an initial Hamiltonian $H_0 = H(\lambda_0)$
- After time t parameter switch to λ_f, such that $H_f = H(\lambda_f)$
- Time evolution of state:

$$|\psi_0(t)\rangle = e^{-iH_f t} |\psi_0\rangle$$

In our problem

$$H_f = \sum_k [\epsilon_k(\eta_k^\dagger \eta_k + \eta_{-k}^\dagger \eta_{-k} - 1)]$$

$$|\psi_{h,\gamma}\rangle = \prod_{k>0} [\cos\Omega_k - i\sin\Omega_k \eta_{-k}^\dagger \eta_k^\dagger] |\psi_{h+\delta,\gamma+\delta}\rangle$$

$$\Omega_k = \frac{1}{2}[\theta_k(h,\gamma) - \theta_k(h+\delta,\gamma+\delta)]$$

- We defined Loschmidt echo

$$\mathcal{L} \equiv \sum_k \log[1 - \sin^2(2\Omega_k)\sin^2(\epsilon_k(h + c_1\delta, \gamma + c_2\delta)t)]$$

- Complexity defined:

$$\mathcal{C}_N(t) = \sum_k \phi_k^2(h + c_1\delta, \gamma + c_2\delta, t)$$

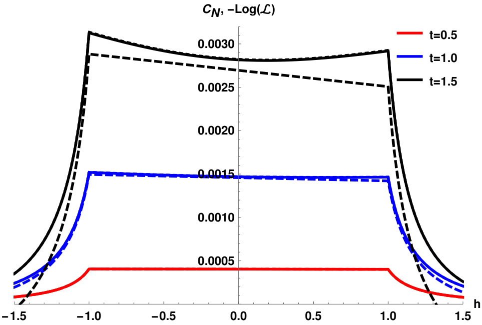
where

$$\phi_k = \arccos[\sqrt{1 - \sin^2(2\Delta\theta_k)\sin^2(\epsilon_k(h + c_1\delta, \gamma + c_2\delta)t)}]$$

- For small time $t \to 0$

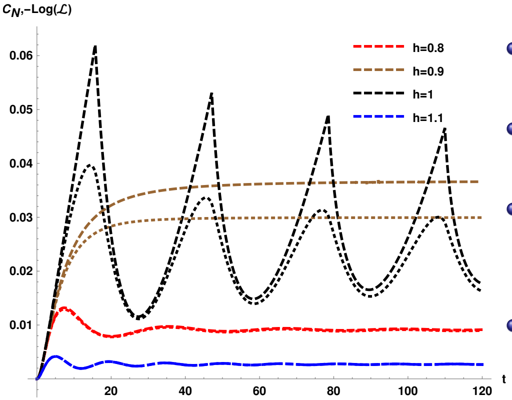
$$\mathcal{L} \approx e^{-\mathcal{C}_N}$$

Small-time behaviour



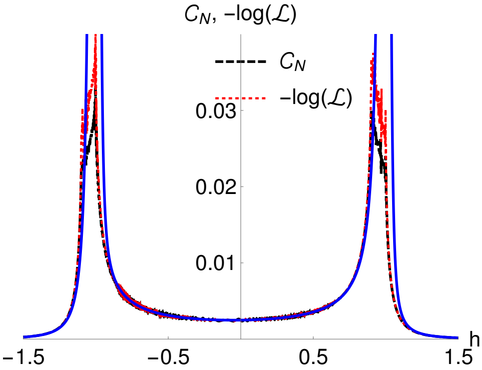
- $c_1 = 1$, $c_2 = 0$: $h \rightarrow h + \delta$
- Discontinuity around the critical points $h = -1$ and $h = 1$
- For small time numerical value of $\mathcal{L}$ and $\mathcal{C}_N$ is indistinguishable from the approximate value obtained by series expansion
- For time $t = 1.5$ analytical expression starts deviating from numerical results

Finite time behaviour



- Dotted lines are complexity, and dashed lines are Loschmidt echo
- Away from the critical point, with time, it gets saturated
- Quenched state at a critical point: temporal oscillation dies out for a large time
- Initial state at a critical point: oscillation continues for large time

Large time behaviour



- These results are valid away from the critical point
- For large time: time-dependent factor will average out and gives $\sin^2(\epsilon t) \approx \frac{1}{2}$
- Series expansion around $\delta = 0$

$$\mathcal{C}_N\big|_{|h|<1}(t \to \infty) = \frac{\delta^2}{8|\gamma|(1-h^2)}$$

$$+O(\delta^3)$$

Four spin interaction model

Hamiltonian

$$H = -h\sum_n(\mu_1 S^z_{n,1} + \mu_2 S^z_{n,2}) - J_1\sum_n(S^x_{n,1}S^x_{n,2} + S^y_{n,1}S^y_{n,2})$$

$$-J_2\sum_n(S^x_{n,2}S^x_{n+1,1} + S^y_{n,2}S^y_{n+1,1}) - J_{13}\sum_n(S^x_{n,1}S^z_{n,2}S^x_{n+1,1} + S^y_{n,1}S^z_{n,2}S^y_{n+1,1})$$

$$-J_{23}\sum_n(S^x_{n,2}S^z_{n+1,1}S^x_{n+1,2} + S^y_{n,2}S^z_{n+1,1}S^y_{n+1,2})$$

$$-J_{14}\sum_n(S^x_{n,1}S^z_{n,2}S^z_{n+1,2}S^x_{n+1,2}$$

$$-J_{24}\sum_n(S^x_{n,2}S^z_{n+1,1}S^z_{n+1,2}S^x_{n+2,1} + S^y_{n,2}S^z_{n+1,1}S^z_{n+1,2}S^y_{n+1,2})$$

- Set of parameters chosen for four spin cases:

$$\mu_1 = 3\mu, \quad \mu_2 = \mu, \quad h = \frac{h}{2}, \quad J_1 = 2J, \quad J_2 = -1$$

$$J_{13} = 5J_3 \quad , J_{23} = J_3, \quad J_{14} = 4 \quad J_{24} = 0$$

- Set of parameters chosen for three spin cases:

$$\mu_1 = \mu_2 = 1 \quad J_{13} = J_{23} = J_3 \quad J_{14} = 0 \quad J_{24} = 0$$

- Hamiltonian in the diagonal form after using Jordan Wigner transformation and Bogoliubov transformation:

$$H = \sum_k (\epsilon_{k,1}\eta^{\dagger}_{k,1}\eta_{k,1} + \epsilon_{k,2}\eta^{\dagger}_{k,2}\eta_{k,2})$$

where dispersion relation

$$\epsilon_{k,1,2} = [h - \frac{3J_3}{2}\cos k] \mp \sqrt{(\frac{h}{2} - J_3 \cos k)^2 + J^2 + \sin^2 k}$$

- Complexity

$$\mathcal{C}_N(t) = \sum_k \phi_k^2(h+\delta, J3)$$

where

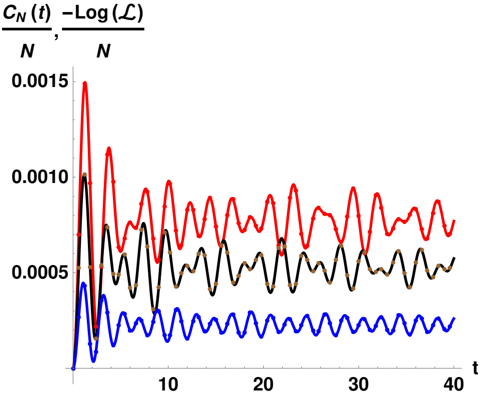
$$\phi_k = \arccos[\sqrt{1-\sin^2(2\Omega_k)\sin^2(\Delta_k(h+\delta, J_3)t)}]$$

$$\Omega_k = \frac{1}{2}[\theta_k(h, J_3) - \theta_k(h+\delta, J_3)]$$

$$\Delta_k(h+\delta, J_3) = \sqrt{(\frac{h}{2} - J_3\cos k)^2 + 1 + \sin^2 k}$$

- Loschmidt Echo:

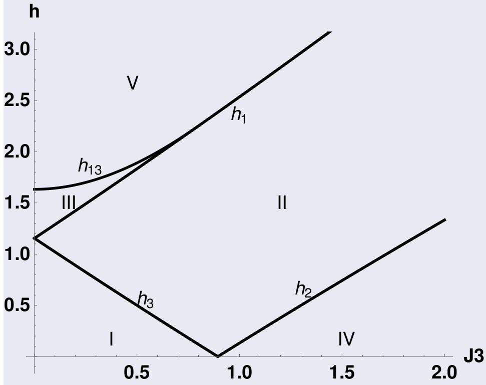
$$\mathcal{L} = \sum_k \log[1-\sin^2(2\Omega_k)\sin^2(\Delta_k(h+\delta, J_3)t)]$$



- Relation $\mathcal{L} \approx e^{-C_N}$ still hold
- Temporal oscillation persists in large time
- No special behaviour around the critical point

Phase Diagram

Four spin



$$h_{1,2} = \frac{1}{3}(\pm\sqrt{12J^2 + J_3^2} + 4J_3)$$

$$h_3 = \frac{1}{3}(\sqrt{12J^2 + J_3^2} + 4J_3)$$

$$h_{13} = 2\sqrt{\frac{(J^2+1)(4+5J_3^2)}{12 - J_3^2}}$$

I :$0 < h < h_3$ for $J_3 < 0.88$ **II** :$h_3 < h$ and $h_1 < h < h_{13}$

III :$h_{13} < h < h_1$ for $J_3 < 0.88$ **IV** :$0 < h < h_2$ for $J_3 > 0.88$

V :$h_{13} < h$ for $J_3 < 0.88$ and $h > h_1$ for $J_3 > 0.88$

Future Direction

- Universal behaviour of Complexity?
- Nielsen complexity does not have an algorithm structure (in the coming project, we are working on Krylov complexity)
- Question of penalty factor? How does it affect complexity?
- Non Riemanian metric ?
- Complexity behaviour under local quench

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