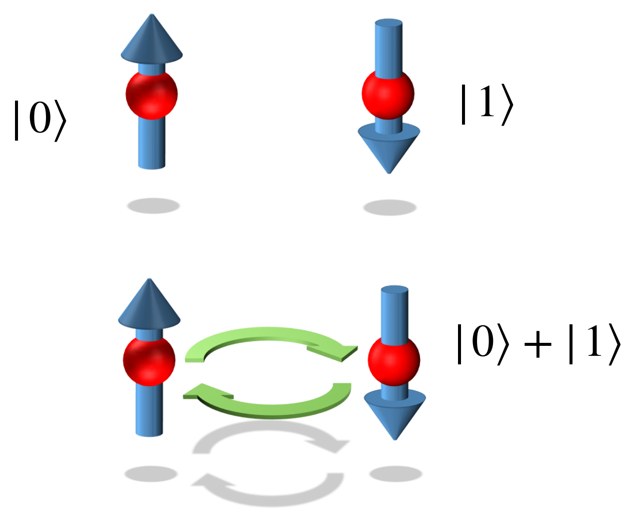
Zahra Raissi

Quantum Information group at Paderborn University

Many-body entanglement and quantum error correction

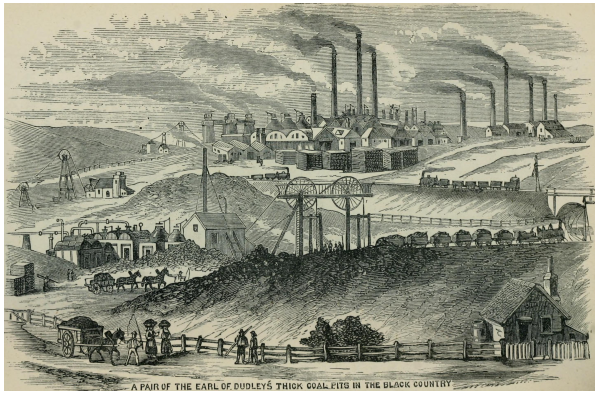
Zahra Raissi

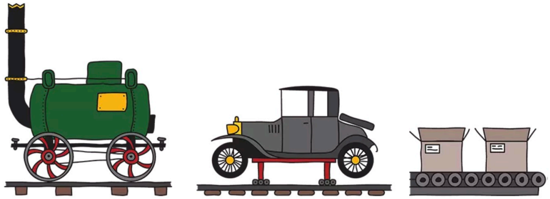
Junior group leader at Paderborn University



Quantum Entanglement in High Energy Physics
12 May, 2023

The industrial revolution through time

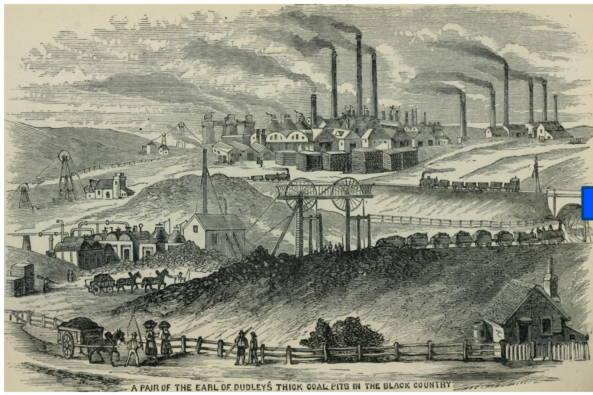


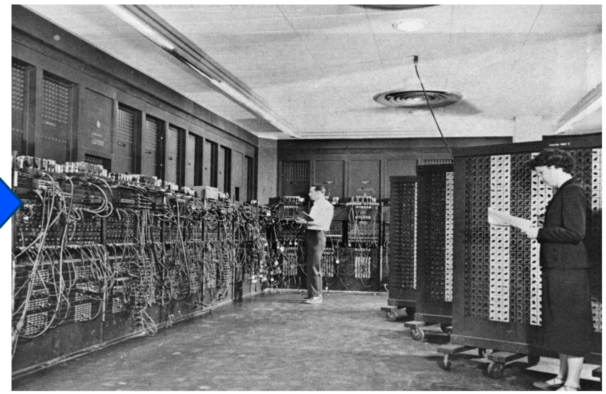


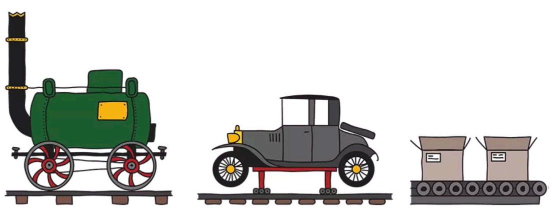
18th - 19th Century

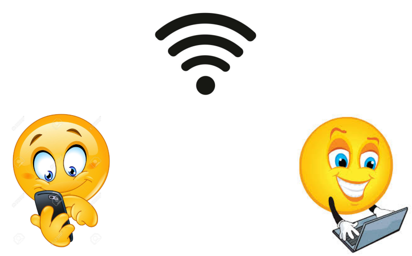
The Industrial Revolution

The industrial revolution through time









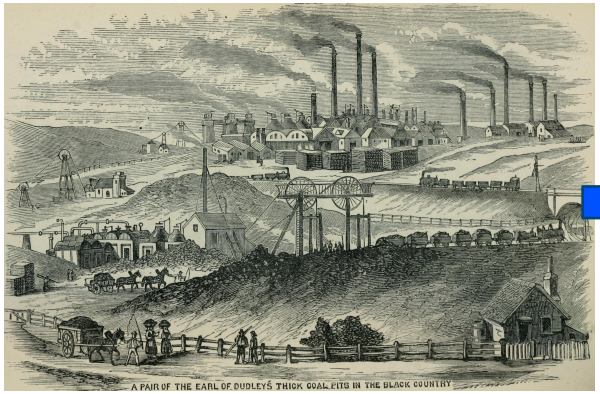
18th - 19th Century

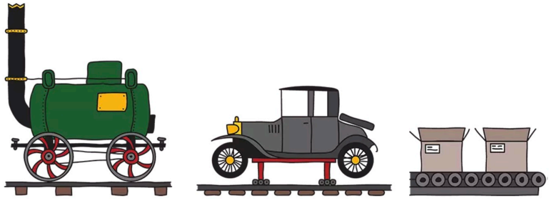
The Industrial Revolution

20th Century

The Information Technology Revolution

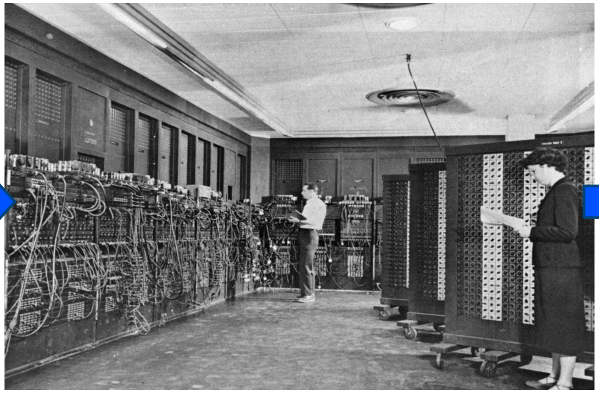
The industrial revolution through time





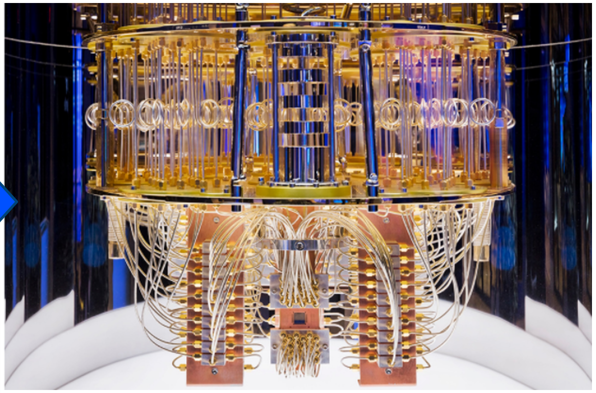
18th - 19th Century

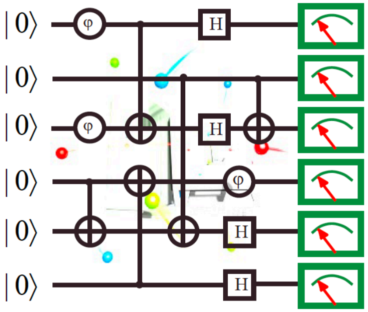
The Industrial Revolution



20th Century

The Information Technology Revolution



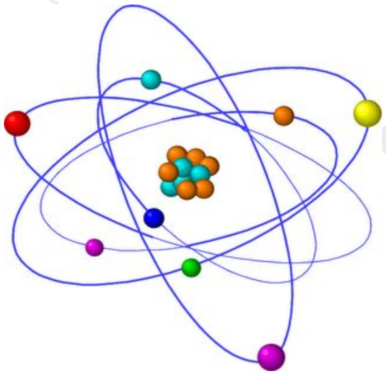


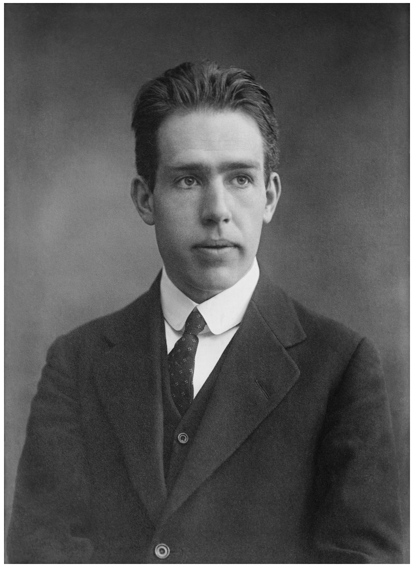
21st Century

The Quantum-Technology Revolution?

Quantum mechanics

- Quantum mechanics is almost 100 years old …





Niels Bohr

spectral lines of

hydrogen atoms (1913)

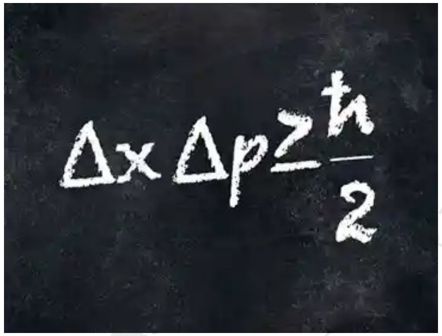
Erwin Schrödinger

wave mechanics (1926)

Werner Heisenberg

matrix mechanics (1925)

uncertainty principle (1927)

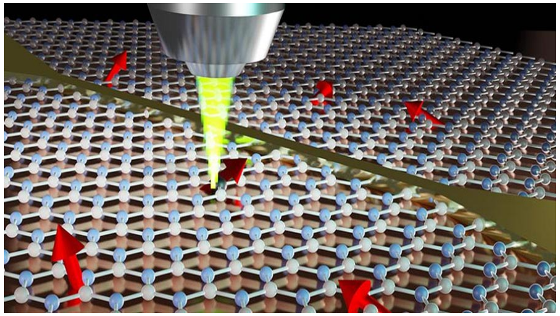


Quantum mechanics

- …. but remains alive and full of challenging open problems.

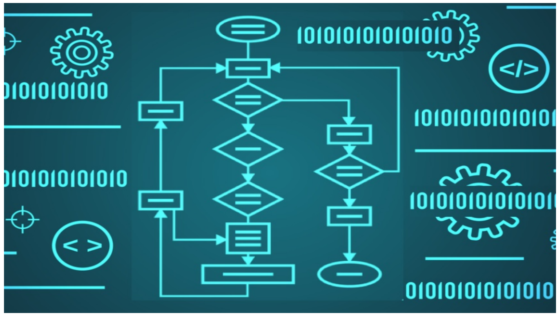
Quantum computing

Quantum data storage



Quantum network

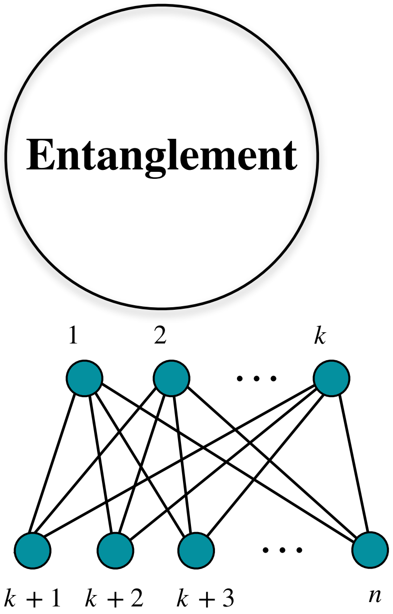
Quantum algorithms



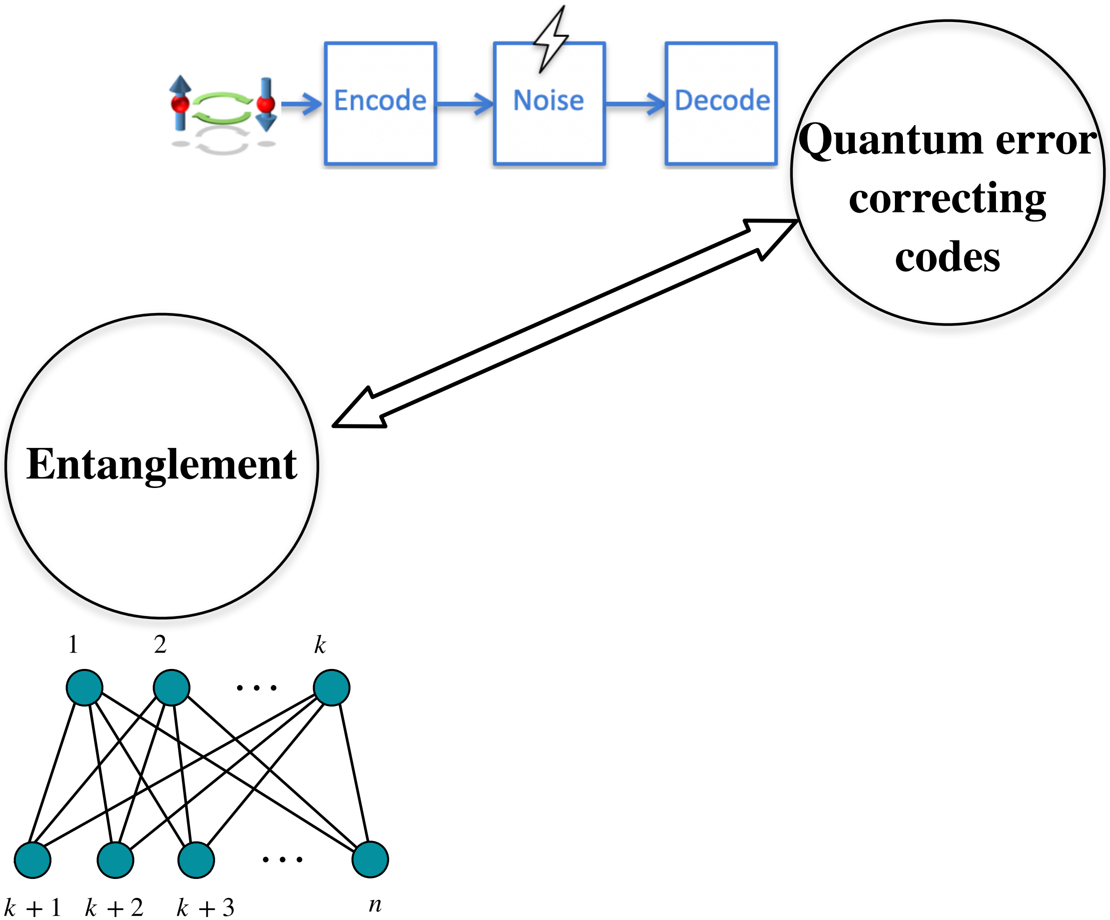
Quantum secret sharing

Teleportation

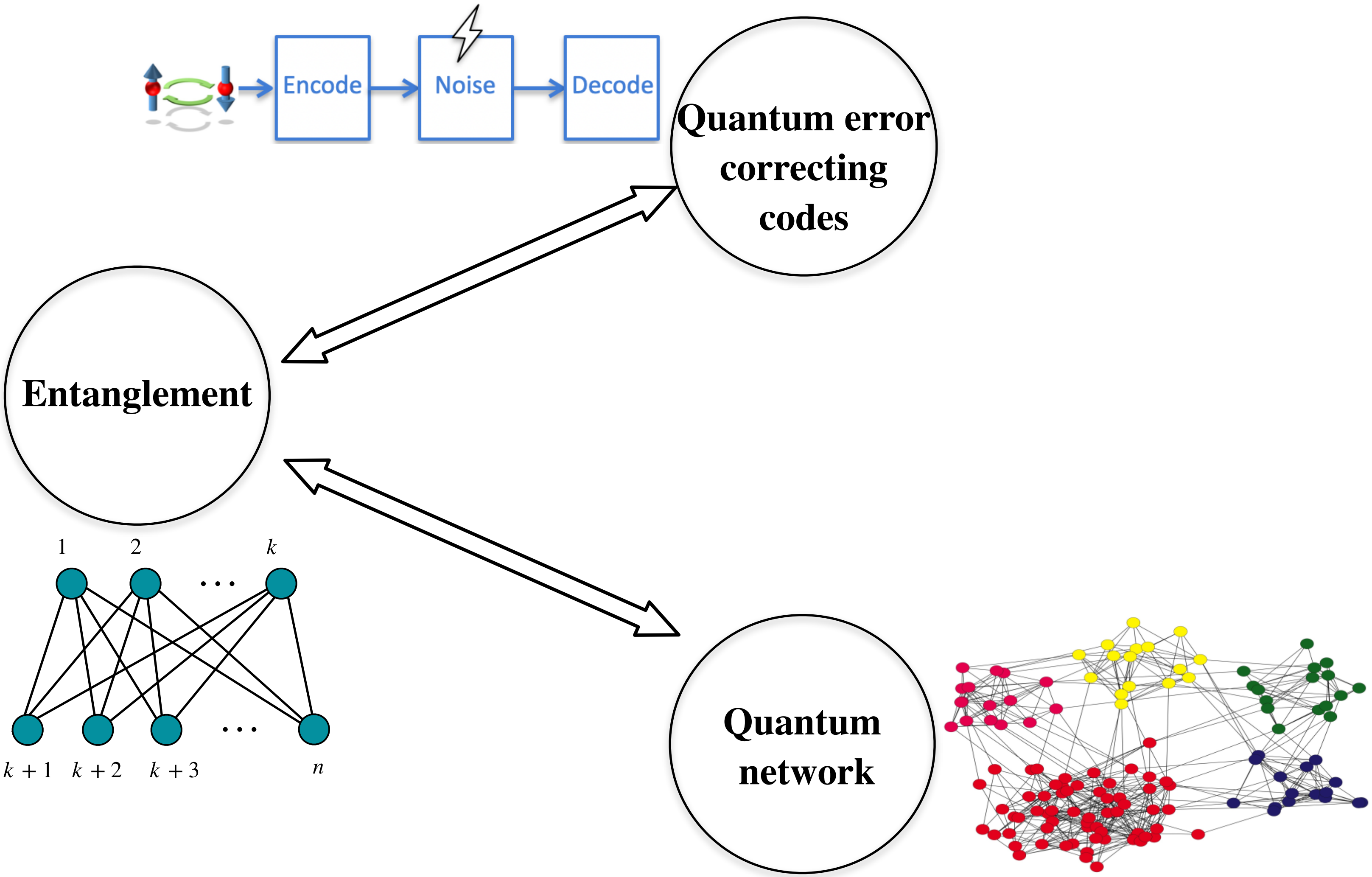
Entanglement in general



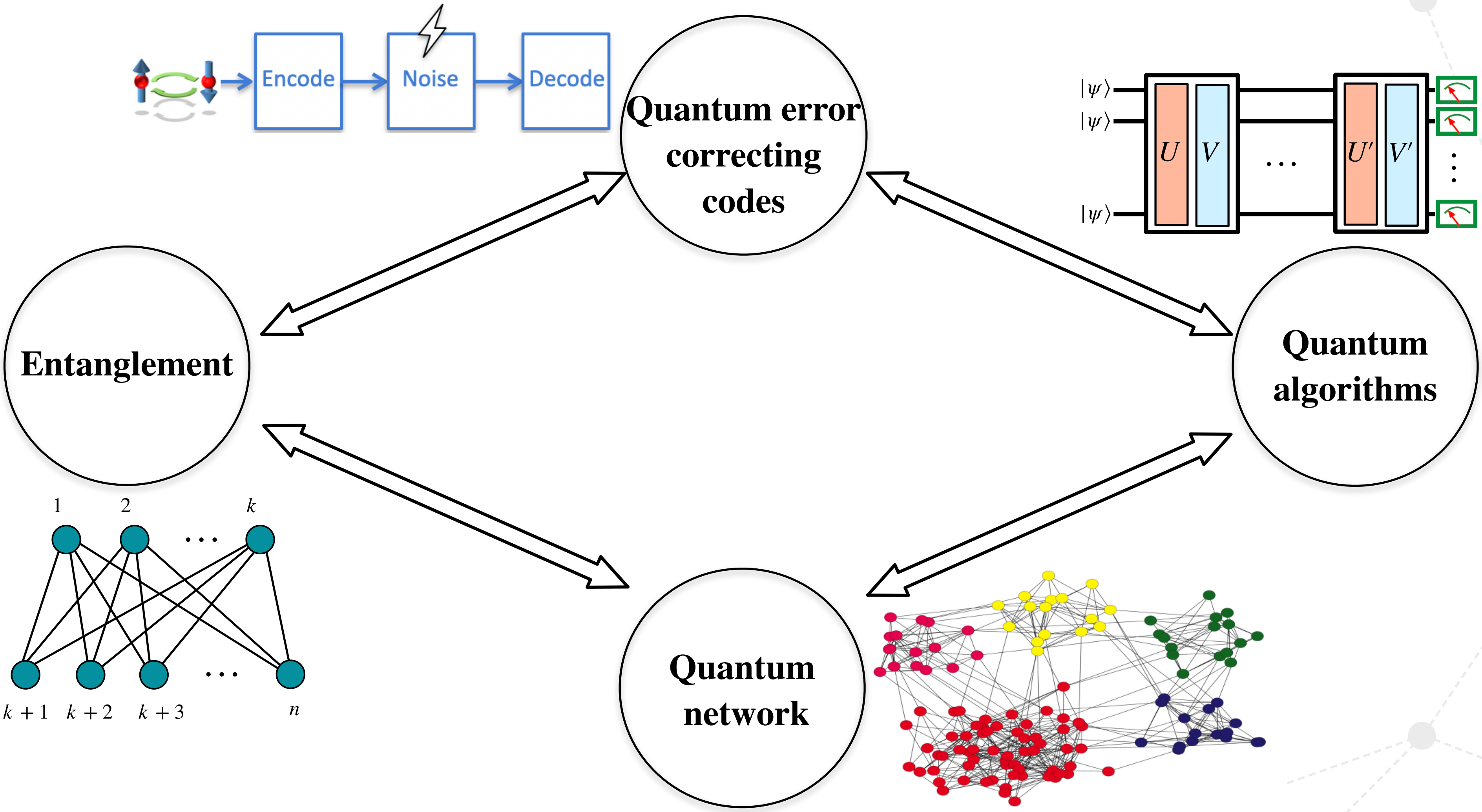
Entanglement in general



Entanglement in general

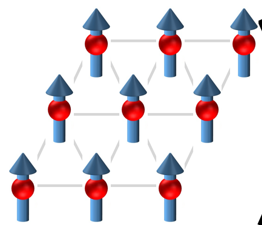
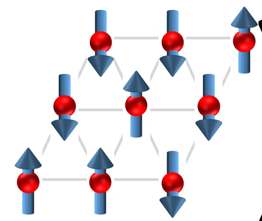
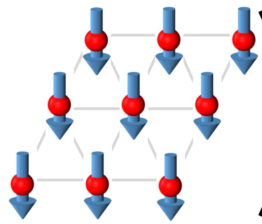


Entanglement in general



Multipartite systems

Quantum many-body system: the system is in a superposition of the possible configurations

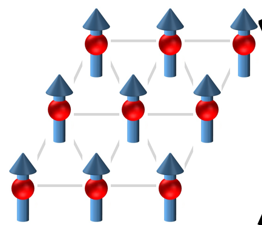
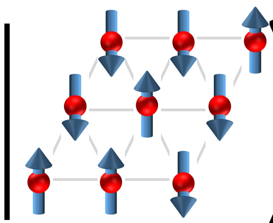
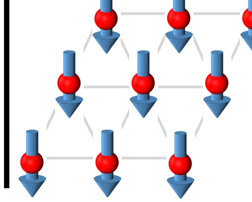
$|\psi_{1,\dots,n}\rangle = C_1$  $+ \dots + C_i$  $+ \dots + C_{2^n}$ 

exponentially many

$$= C_1 |0000\dots0\rangle + \dots + C_i |1101\dots1\rangle + \dots + C_{2^n} |1111\dots1\rangle$$

Multipartite systems

Quantum many-body system: the system is in a superposition of the possible configurations

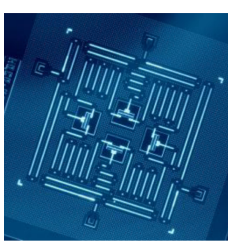
$|\psi_{1,\ldots,n}\rangle = C_1 \Big|$  $\Big\rangle + \ldots + C_i \Big|$  $\Big\rangle + \ldots + C_{2^n} \Big|$  $\Big\rangle$

exponentially many

$$= C_1|0000\ldots0\rangle + \ldots + C_i|1101\ldots1\rangle + \ldots + C_{2^n}|1111\ldots1\rangle$$

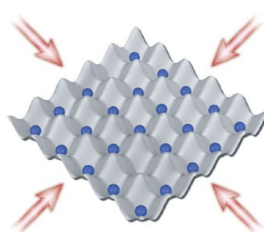
Today we have quantum simulators to study many-body systems:

Superconducting qubits



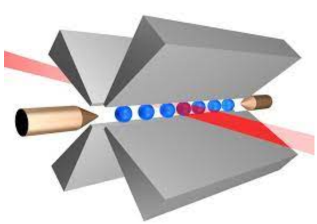
@Companies such as Google, IBM, …

Optical lattices



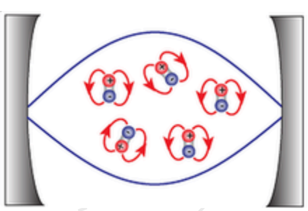
@MPQ, …

Trapped ions



@UIBK (Innsbruck), JQI (Maryland),…

Cavity quantum electrodynamics



@TU Wien,…

Outline

- Entanglement: resource to overcome LOCC
  - Two-body (bipartite) systems
  - Many-body (multipartite) systems
- Well known many-body entangled classes
- Quantum error correcting codes

Outline

- Entanglement: resource to overcome LOCC
  - Two-body (bipartite) systems
    - What is known about two-body entanglement?
    - How to classify the states?
    - What is the maximally entangled state?
  - Many-body (multipartite) systems
- Well known many-body entangled classes
- Quantum error correcting codes

Entanglement

Bob

(Poland)

qubit

qubit

Alice

(Germany)

Entanglement

Bob

(Poland)

qubit

qubit

Alice

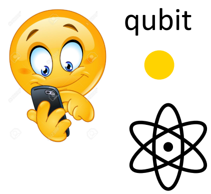
(Germany)

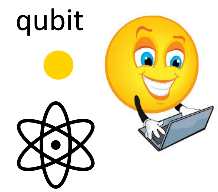
What can they do?

Entanglement

Bob

(Poland)





Alice

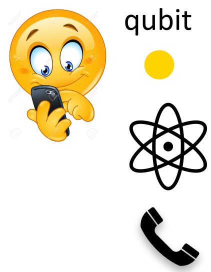
(Germany)

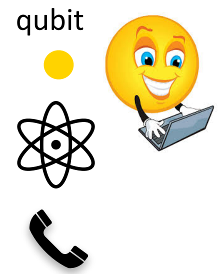
What can they do? Apply Local Operations

Entanglement

Bob

(Poland)





Alice

(Germany)

What can they do? Apply Local Operations and Classical Communications

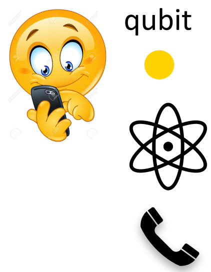
Entanglement

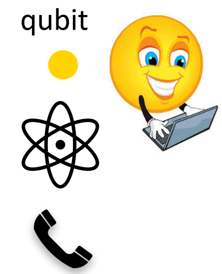
What can they do? Apply Local Operations and Classical Communications } LOCC

Entanglement

Bob

(Poland)





Alice

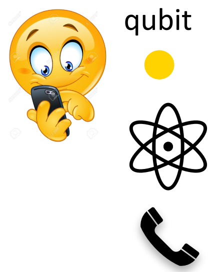
(Germany)

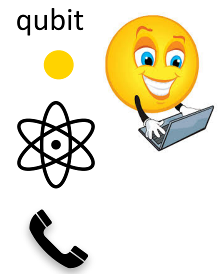
What can they do? Apply **L**ocal **O**perations and **C**lassical **C**ommunications } LOCC

What do we want to do?

Entanglement

Bob
(Poland)





Alice
(Germany)

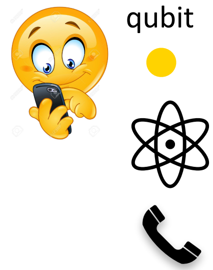
What can they do? Apply Local Operations and Classical Communications } LOCC

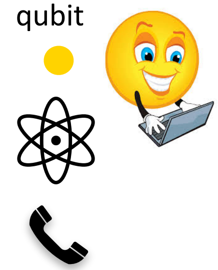
What do we want to do?

- Teleportation 

Entanglement

Bob
(Poland)





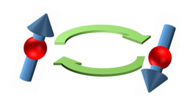
Alice
(Germany)

What can they do? Apply Local Operations and Classical Communications } LOCC

What do we want to do?

- Teleportation

time & space →

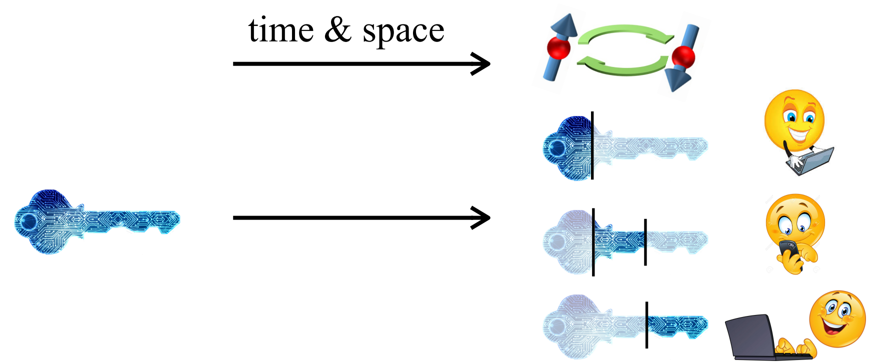


Entanglement

What can they do? Apply **L**ocal **O**perations and **C**lassical **C**ommunications } LOCC

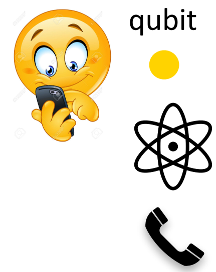
What do we want to do?

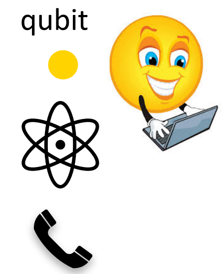
- Teleportation
- Secret sharing



Entanglement

Bob
(Poland)





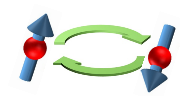
Alice
(Germany)

What can they do? Apply Local Operations and Classical Communications } LOCC

What do we want to do?

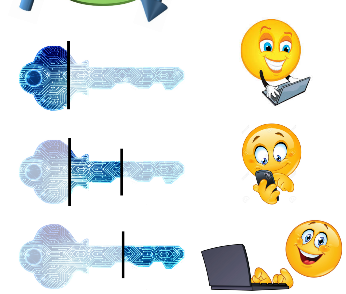
- Teleportation

time & space →



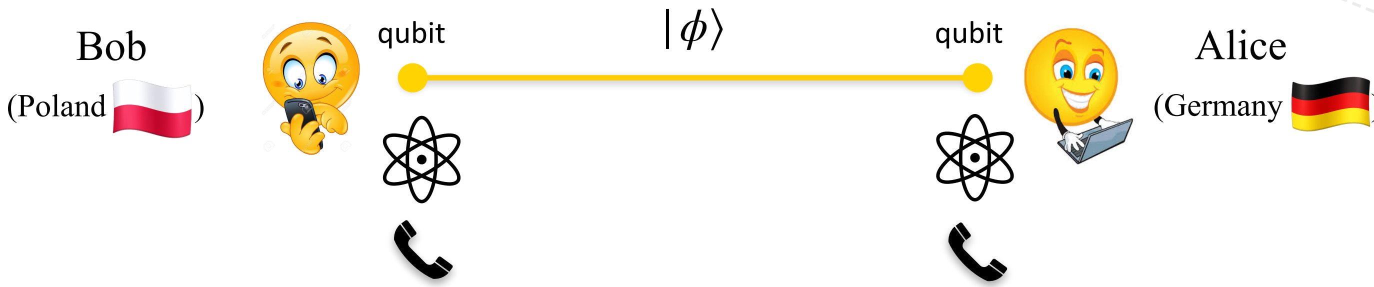
- Secret sharing

→



- Entanglement based Q. Communication

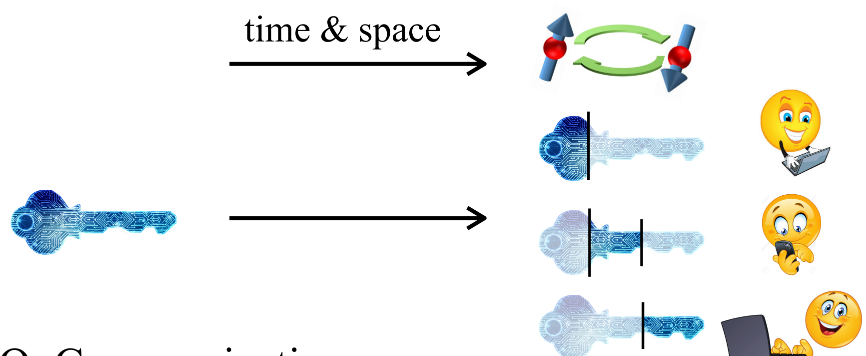
Entanglement



What can they do? Apply Local Operations and Classical Communications } LOCC

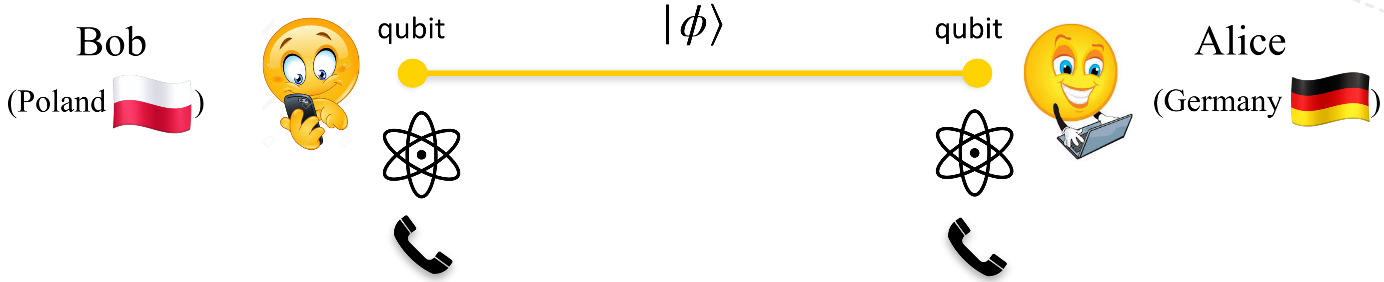
What do we want to do?

- Teleportation — time & space →
- Secret sharing →
- Entanglement based Q. Communication



- Use entanglement as a resource.

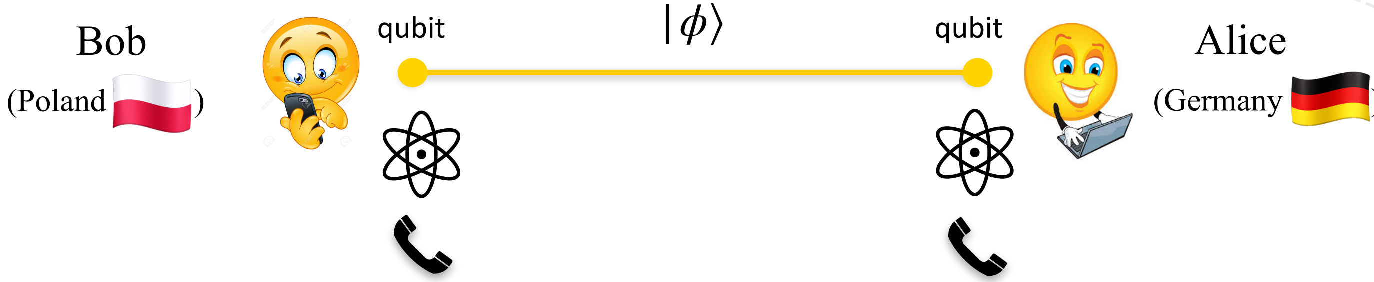
Classification of entanglement



$$|\phi\rangle \overset{\text{LOCC}}{\rightleftarrows} |\psi\rangle$$

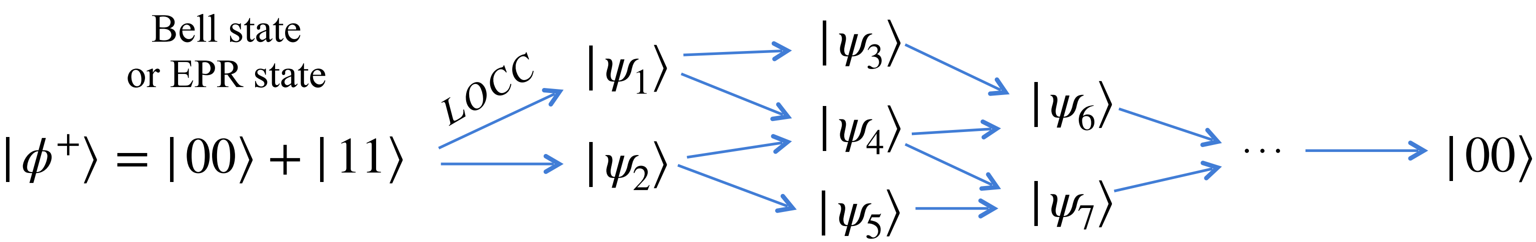
Whatever Alice and Bob can do using $|\phi\rangle \longrightarrow$ they can do using $|\psi\rangle$

Classification of entanglement



$$|\phi\rangle \underset{}{\overset{\text{LOCC}}{\rightleftarrows}} |\psi\rangle$$

Whatever Alice and Bob can do using $|\phi\rangle$ ⟶ they can do using $|\psi\rangle$



✓ $|\phi^+\rangle$ is maximally entangled state

$$E(|\phi^+\rangle) \geq E(|\psi_i\rangle) \quad \forall i$$

- E is any entanglement measure ⟶ only condition: non-increasing under LOCC

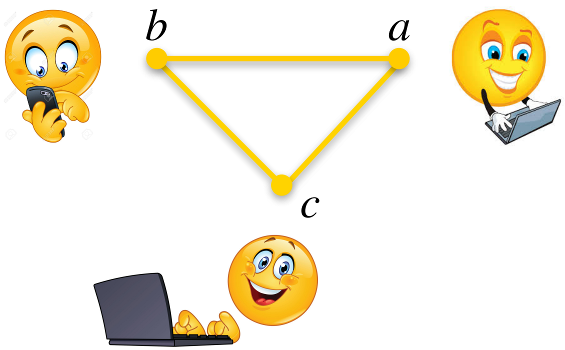
Outline

- Entanglement: resource to overcome LOCC
  - Two-body (bipartite) systems
  - Many-body (multipartite) systems
- Well known many-body entangled classes
- Quantum error correcting codes

Outline

- Entanglement: resource to overcome LOCC
  - Two-body (bipartite) systems
  - Many-body (multipartite) systems
    - What is known about multipartite entangled states?
    - What makes multipartite different from bipartite entanglement?
    - What makes classification of multipartite entanglement difficult?
- Well known many-body entangled classes
- Quantum error correcting codes

Three parties



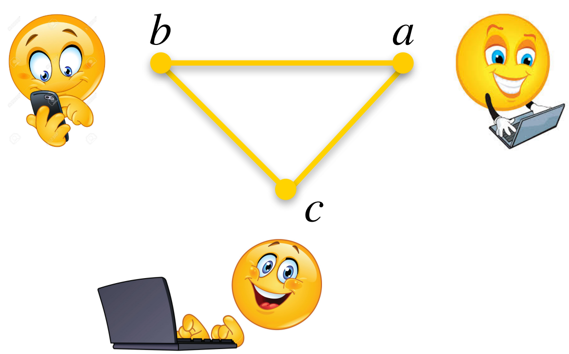
Entangled if $|\psi\rangle \neq |a\rangle \otimes |b\rangle \otimes |c\rangle$

$$|GHZ\rangle = |000\rangle + |111\rangle$$

$$|W\rangle = |001\rangle + |010\rangle + |100\rangle$$

W. Dür, G. Vidal, J. I. Cirac, "Three qubits can be entangled in two inequivalent ways", Phys. Rev. A **62**, 062314 (2000).

Three parties



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$$|GHZ\rangle = |000\rangle + |111\rangle$$

$$|W\rangle = |001\rangle + |010\rangle + |100\rangle$$

Not even probabilistically! $\quad |GHZ\rangle \overset{\text{SLOCC}}{\not\rightleftarrows} |W\rangle$

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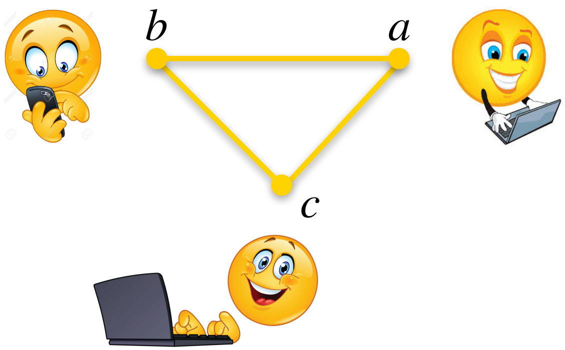


Entangled if $|\psi\rangle \neq |a\rangle \otimes |b\rangle \otimes |c\rangle$

$$|GHZ\rangle = |000\rangle + |111\rangle$$

$$|W\rangle = |001\rangle + |010\rangle + |100\rangle$$

Not even probabilistically! $\quad |GHZ\rangle \overset{\text{SLOCC}}{\not\rightleftarrows} |W\rangle$

There cannot exist one single maximally entangled state.

- GHZ class
- W class

$|GHZ\rangle$

$|\psi_1\rangle \quad |\psi_2\rangle$

$|\psi_3\rangle \quad |\psi_4\rangle \quad |\psi_5\rangle \quad |\psi_6\rangle$

$|W\rangle$

$|\psi_1'\rangle \quad |\psi_2'\rangle$

$|\psi_3'\rangle \quad |\psi_4'\rangle \quad |\psi_5'\rangle \quad |\psi_6'\rangle$

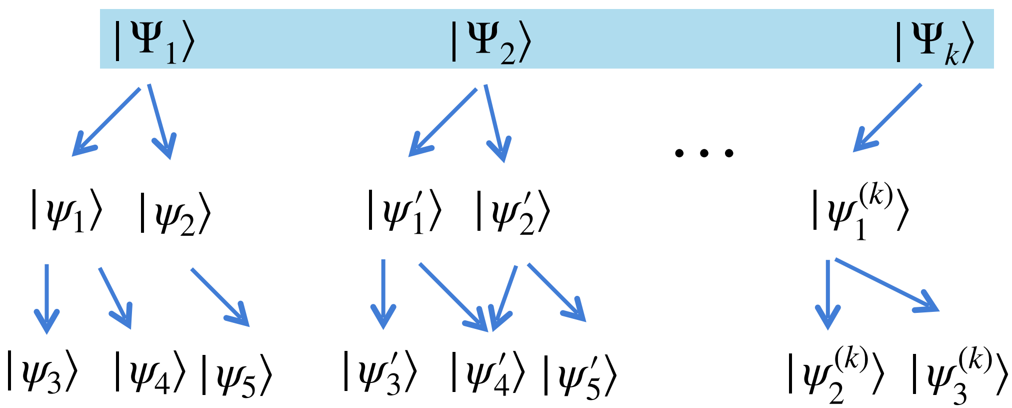
$|000\rangle$

W. Dür, G. Vidal, J. I. Cirac, "Three qubits can be entangled in two inequivalent ways", Phys. Rev. A **62**, 062314 (2000).

Many-body systems

Does a maximally entangled set exist for n-partite states?

$$S_n = \{\, |\Psi_1\rangle, |\Psi_2\rangle, \ldots, |\Psi_k\rangle \}$$

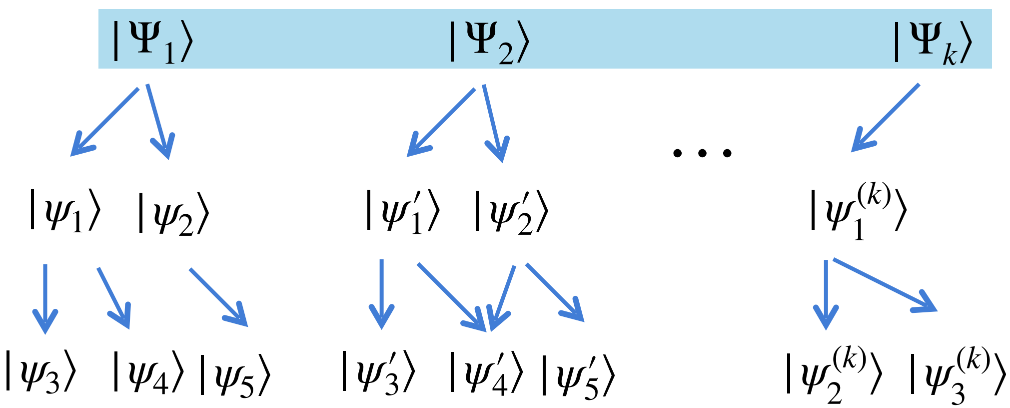


J. I. de Vicente, C. Spee, and B. Kraus, "Maximally Entangled Set of Multipartite Quantum States", PRL. **111**, 110502 (2013).

Many-body systems

Does a maximally entangled set exist for n-partite states?

$$S_n = \{ |\Psi_1\rangle, |\Psi_2\rangle, \ldots, |\Psi_k\rangle \}$$



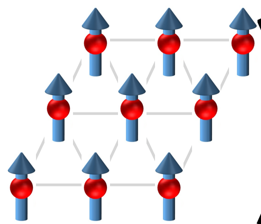
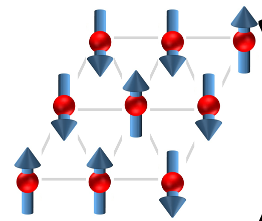
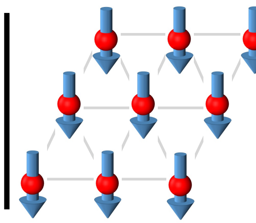
$\boxed{n \geq 4}$

For $n \geq 4$ almost all states are in the maximally entangled set

J. I. de Vicente, C. Spee, and B. Kraus, "Maximally Entangled Set of Multipartite Quantum States", PRL. **111**, 110502 (2013).

Many-body systems

Quantum many-body system: the system is in a superposition of the possible configurations

$|\psi_{1,\ldots,n}\rangle = C_1 \Big|$  $\Big\rangle + \ldots + C_i \Big|$  $\Big\rangle + \ldots + C_{2^n} \Big|$  $\Big\rangle$

exponentially many

$$= C_1|0000\ldots0\rangle + \ldots + C_i|1101\ldots1\rangle + \ldots + C_{2^n}|1111\ldots1\rangle$$

Many-body systems

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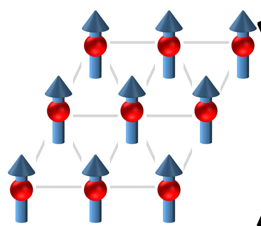
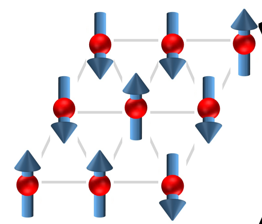
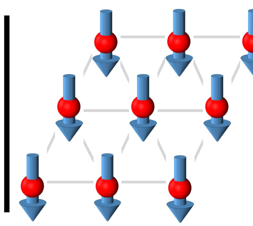
$|\psi_{1,\ldots,n}\rangle = C_1 \Big|$  $\Big\rangle + \ldots + C_i \Big|$  $\Big\rangle + \ldots + C_{2^n} \Big|$  $\Big\rangle$

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What do we want to know?

Many-body systems

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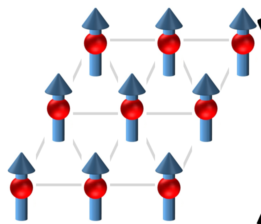
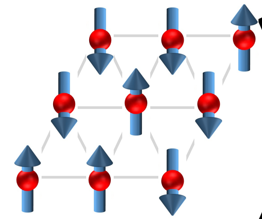
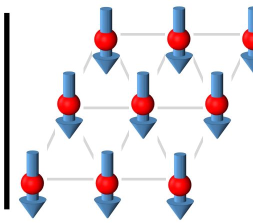
$|\psi_{1,\dots,n}\rangle = C_1 \Big|$  $\Big\rangle + \dots + C_i \Big|$  $\Big\rangle + \dots + C_{2^n} \Big|$  $\Big\rangle$

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$$= C_1 |0000\dots0\rangle + \dots + C_i |1101\dots1\rangle + \dots + C_{2^n} |1111\dots1\rangle$$

What do we want to know?

⇒ What are the most useful states for a given quantum application?

- How powerful is a state?
- What kind of manipulation is possible for the states?
- How to find/construct these states?

Outline

- Entanglement: resource to overcome LOCC
  - Two-body (bipartite) systems
  - Many-body (multipartite) systems

- Well known many-body entangled classes

- Quantum error correcting codes

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- Entanglement: resource to overcome LOCC
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  - *k*-uniform and absolutely maximally entangled (AME) states
  - Graph states
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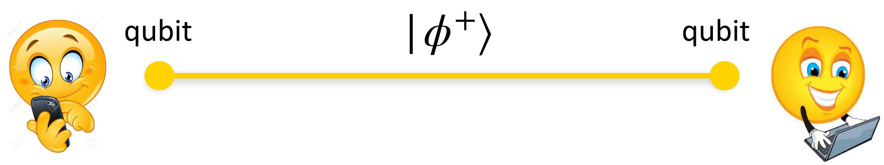
Multipartite entanglement

In many cases we are looking for the most useful states for a given quantum application.

Multipartite entanglement

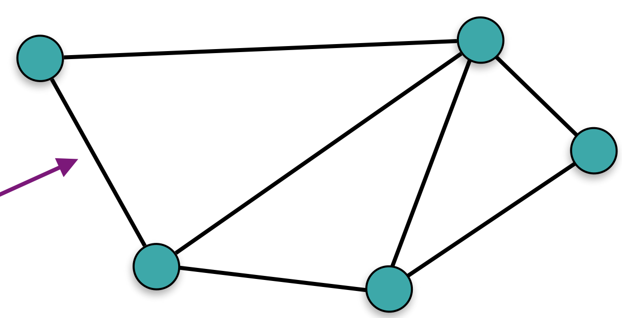
⇨ In many cases we are looking for the most useful states for a given quantum application.

- *k*-uniform and Absolutely Maximally Entangled (AME) states:



Generalise Bell state to an arbitrary number of parties $|\phi^+\rangle = |00\rangle + |11\rangle \longrightarrow \begin{cases} \rho_A \propto I \\ \rho_B \propto I \end{cases}$

- Graph states:

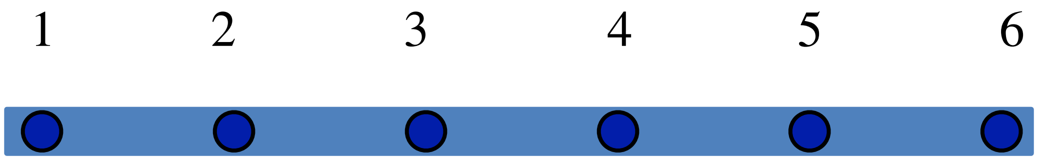


Entangling gate
(Controlled-*Z* operator)

qubit
or
qudit

What are k-uniform states?

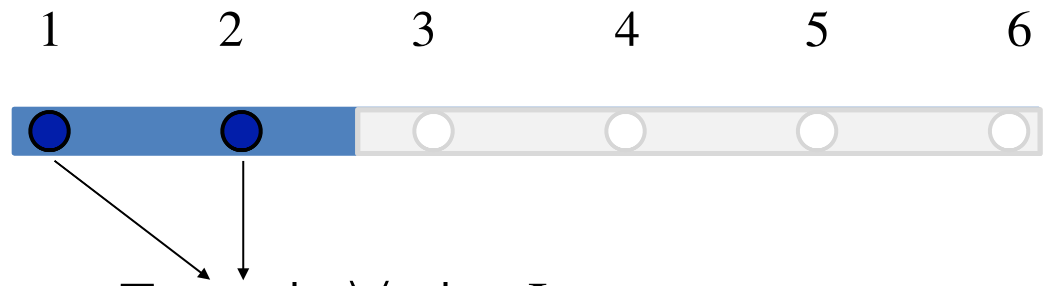
$$|\psi\rangle \;=$$



What are k-uniform states?

2 uniform state

$|\psi\rangle =$

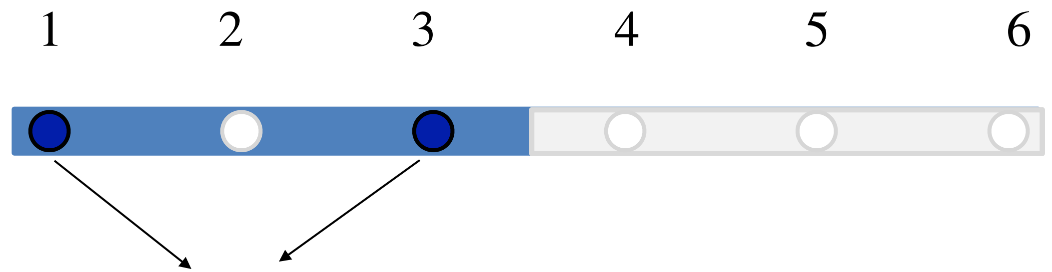


$$\mathrm{Tr}_{\{1,2\}^c}|\psi\rangle\langle\psi| \propto \mathrm{I}$$

What are k-uniform states?

2 uniform state

$|\psi\rangle =$

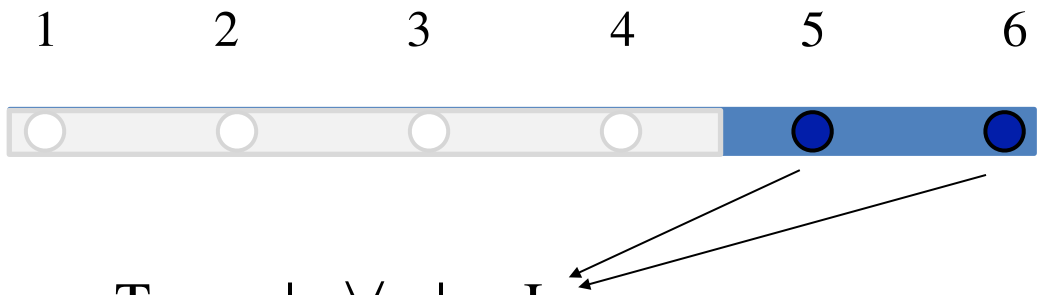


$$\mathrm{Tr}_{\{1,3\}^c}|\psi\rangle\langle\psi| \propto \mathrm{I}$$

What are k-uniform states?

2 uniform state

$$|\psi\rangle =$$



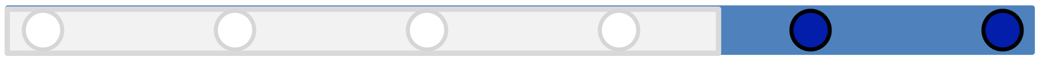
$$\mathrm{Tr}_{\{5,6\}^c} |\psi\rangle\langle\psi| \propto \mathrm{I}$$

For any splitting of the parties into two groups

What are k-uniform states?

2 uniform state

1 2 3 4 5 6

$|\psi\rangle =$ 

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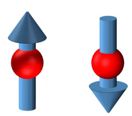
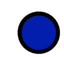
For any splitting of the parties into two groups

qubit 

$|1\rangle$

$|0\rangle$

qudit 

$|q-1\rangle$

$\vdots$

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$|0\rangle$

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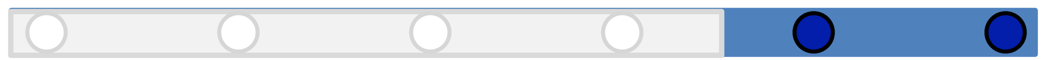
$|\psi\rangle \; =$ 

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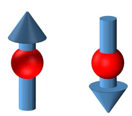
qubit  $|1\rangle$ $|0\rangle$

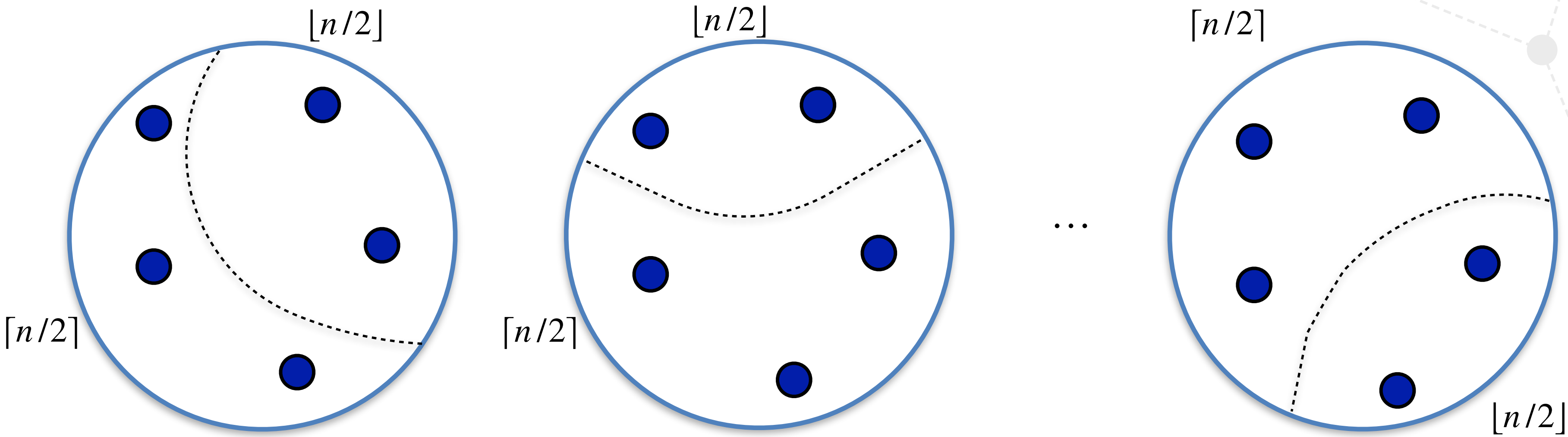
qudit $|q-1\rangle$ $\vdots$ $|1\rangle$ $|0\rangle$

A quantum state of n parties with local dimension q is k-uniform if for all $S \subset \{1, \ldots, n\}$

$$|S| \leq k \quad \Rightarrow \quad \mathrm{Tr}_{S^c}|\psi\rangle\langle\psi| \propto \mathrm{I}$$

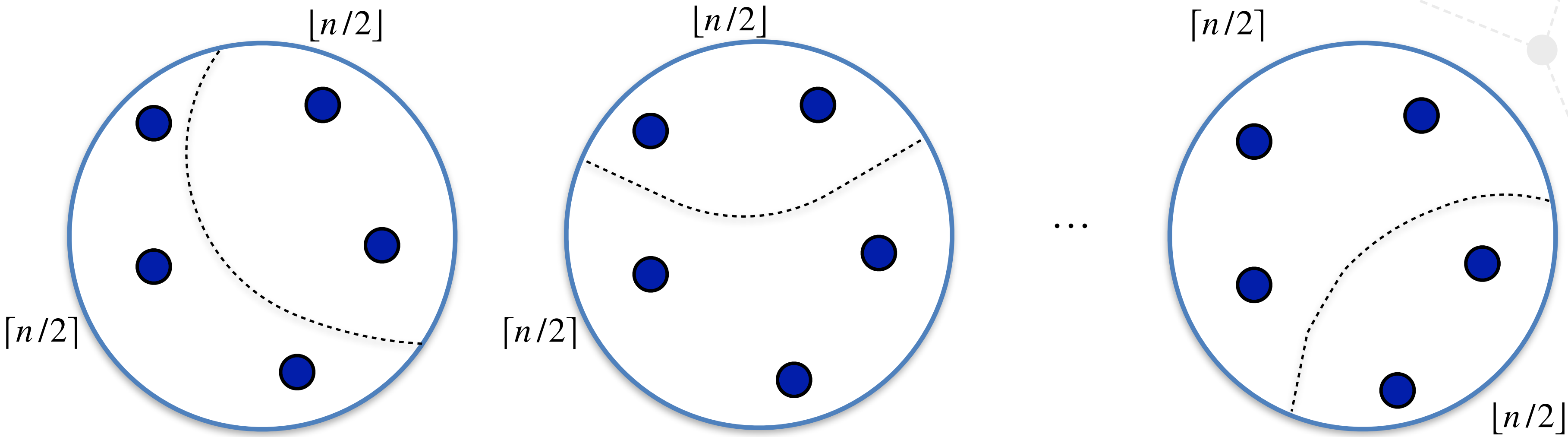
- k-uniform states $\longrightarrow$ k-uniform(n, q)

What are AME states?



For any splitting of the parties into two groups with sizes $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$

What are AME states?



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AME(n, q) states:

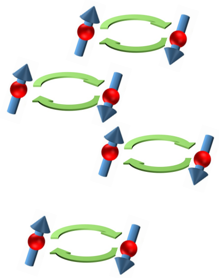
A quantum state of n parties with local dimension q is AME if for all $S \subset \{1, \ldots, n\}$

$$|S| \leq \lfloor n/2 \rfloor \quad \Rightarrow \quad \mathrm{Tr}_{S^c} |\psi\rangle\langle\psi| \propto \mathrm{I}$$

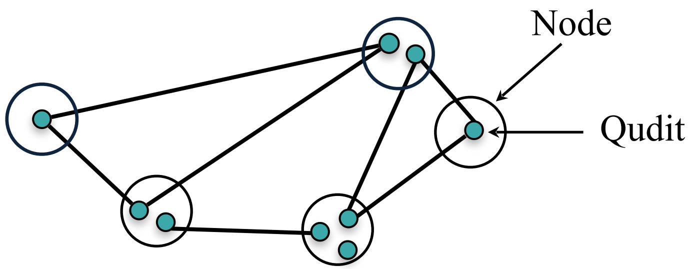
- Absolutely Maximally Entangled (AME) states $\longrightarrow$ AME(n, q)

What are applications?

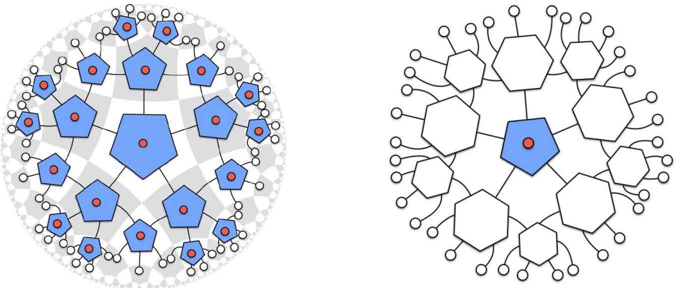
Multipartite teleportation



Quantum networks

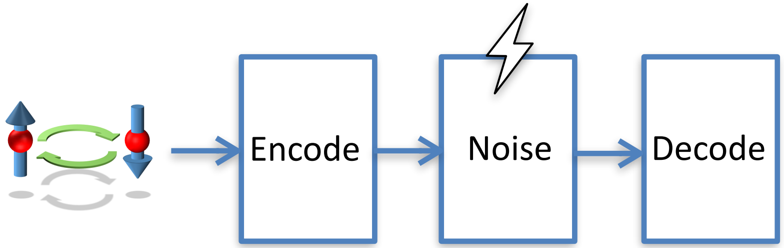


AdS/CFT correspondence [1]



AME states ⟷ Perfect tensors

Quantum error correcting codes

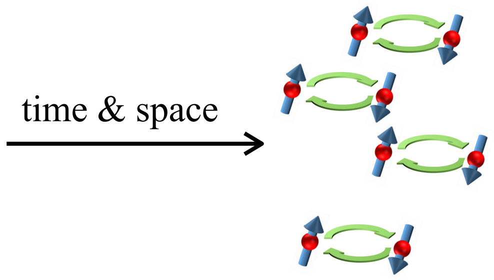


Apart from fundamental point of view of finding maximally multipartite entangled states

[1] F. Patawski, B. Yoshida, D. Harlow, and J. Preskill, Journal of High Energy Physics 06, 149 (2015).

What are applications?

- Multipartite teleportation



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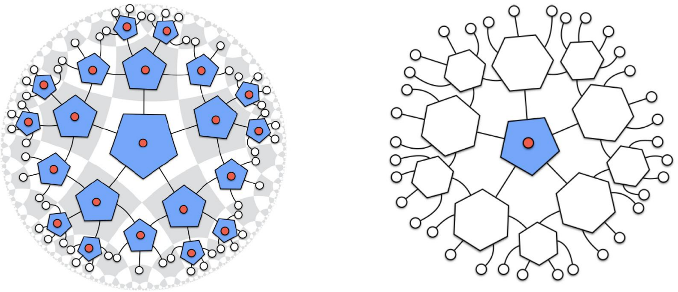

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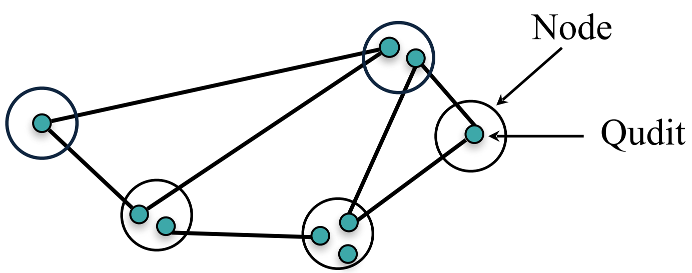

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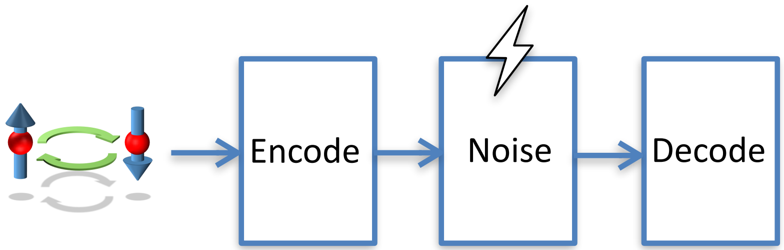

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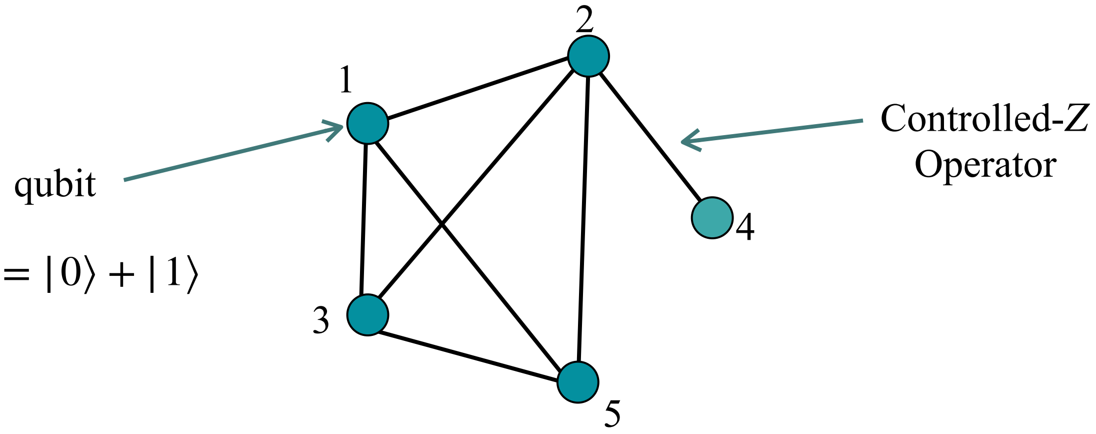
Graph states

- Graph states are pure quantum states that can be represented by a graph.



$$|+\rangle = |0\rangle + |1\rangle$$

$$CZ|00\rangle = |00\rangle$$
$$CZ|01\rangle = |01\rangle$$
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qubit

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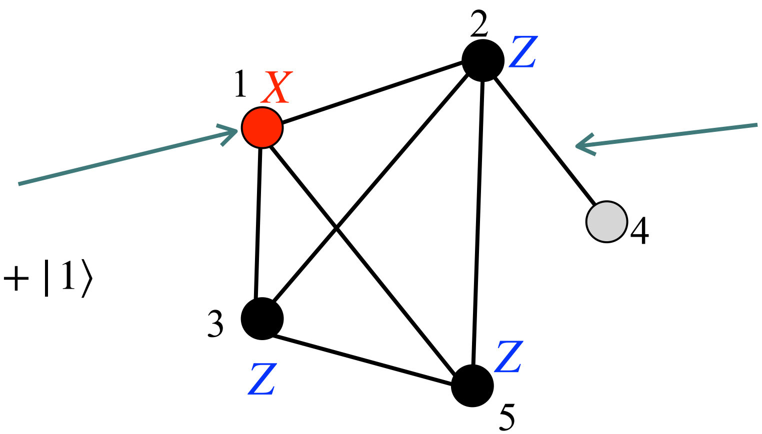

Controlled-Z
Operator

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- Stabilizers:

$$S_1 = \underline{X} \otimes \underline{Z} \otimes \underline{Z} \otimes I \otimes \underline{Z}$$

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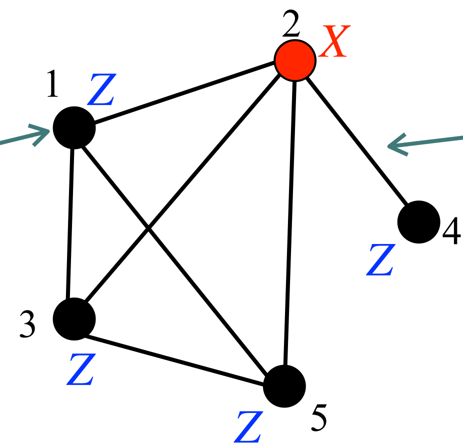

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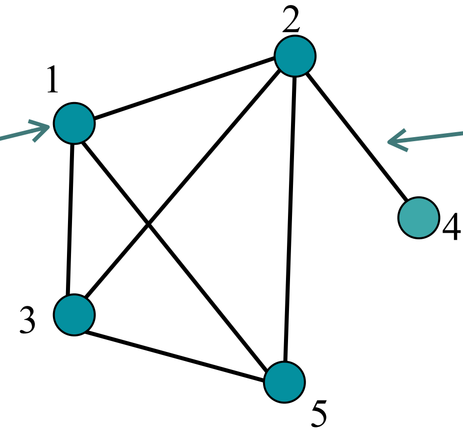

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$$S_4 = I \otimes Z \otimes I \otimes X \otimes I$$

$$S_5 = Z \otimes Z \otimes Z \otimes I \otimes X$$

Graph states

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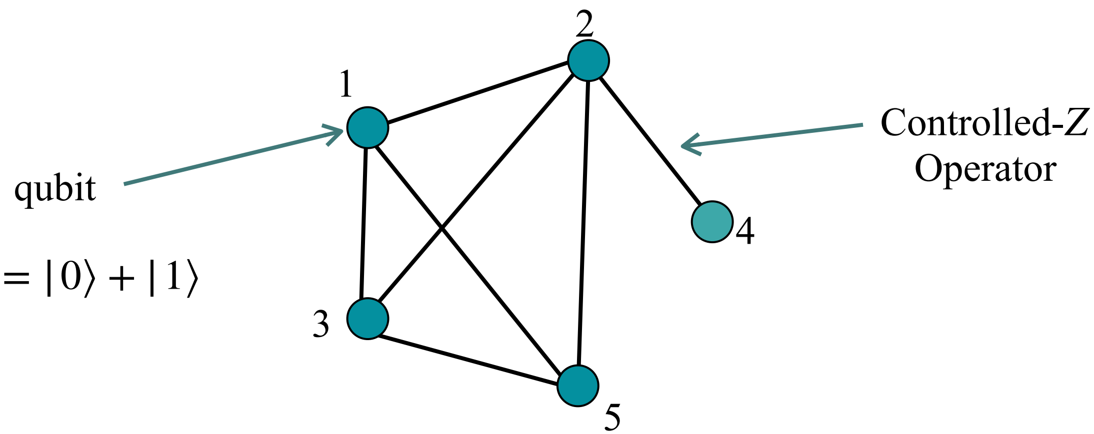

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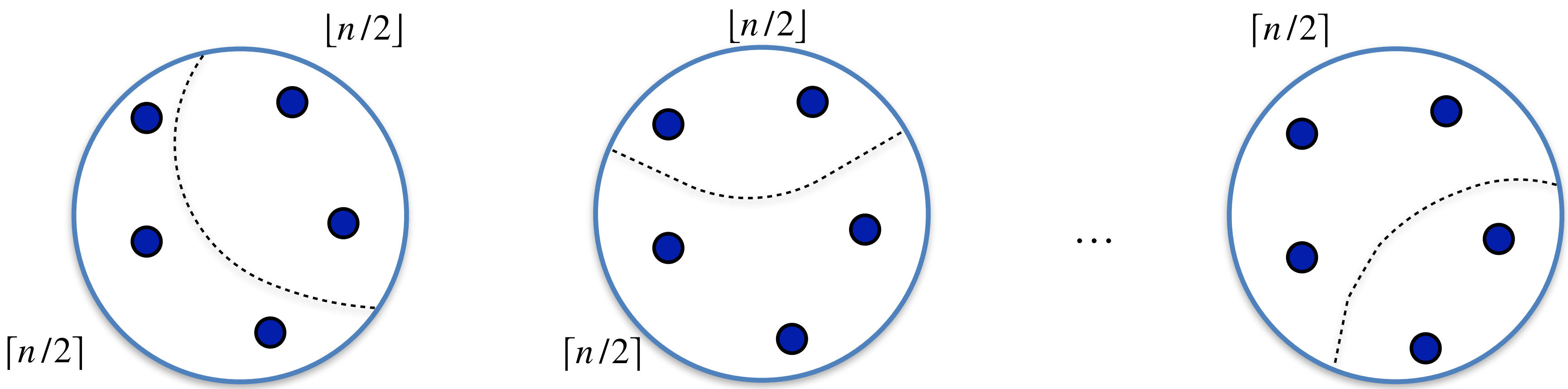
$$S_3 = Z \otimes Z \otimes X \otimes I \otimes Z$$
$$S_4 = I \otimes Z \otimes I \otimes X \otimes I$$
$$S_5 = Z \otimes Z \otimes Z \otimes I \otimes X$$

- Adjacency matrix $\Gamma_{n\times n} \longrightarrow$ symmetric square matrix used to represent a graph.

$$\Gamma = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

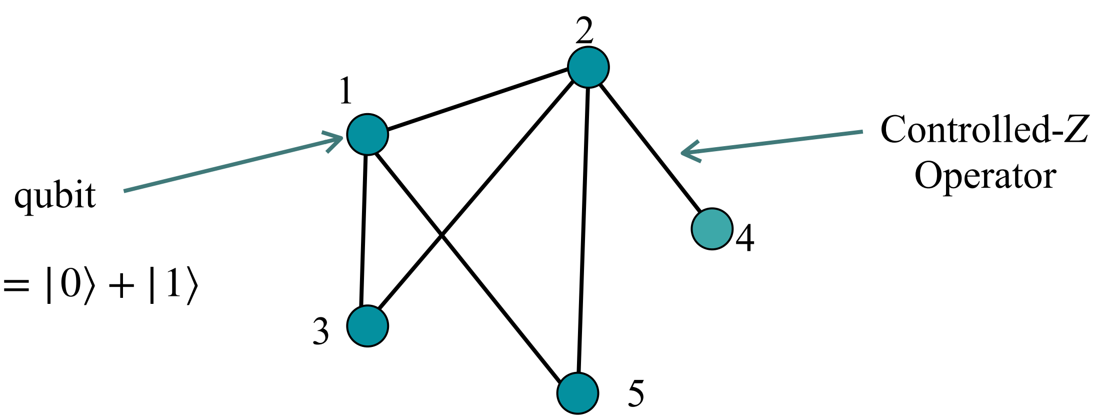
Well known many-body entangled classes

- k-uniform and AME states:



For any splitting of the parties into two groups with sizes $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$

- Graph states:



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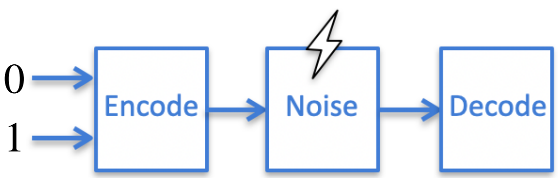
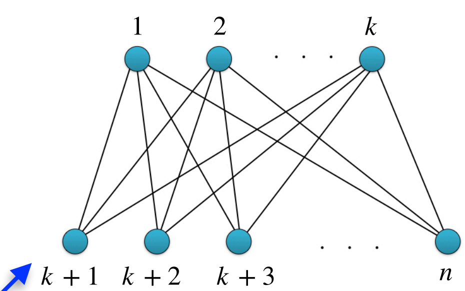
Relation between three concepts

Classical error correcting codes ⇒ ***k*-uniform and AME states** ⇒ **Graph states**



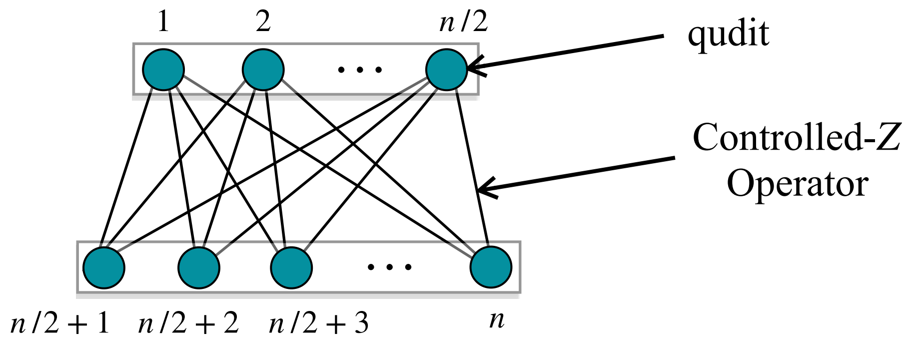


$$|\phi\rangle = \sum |\text{all the codewords}\rangle = \sum_i |\vec{c}_i\rangle$$

ZR, C. Gogolin, A. Riera, A. Acín, J Phys. A: Math. and Theor., 51, 075301 (2018), Highlights of 2018 collection.
ZR, A. Teixidó, C. Gogolin, A. Acín, Phys. Rev. Research **2**, 033411 (2020), arXiv:1910.12789 [quant-ph]

Our results

- The resulting state is a graph state corresponding to a complete bipartite graph.



Classical error correcting codes → AME states

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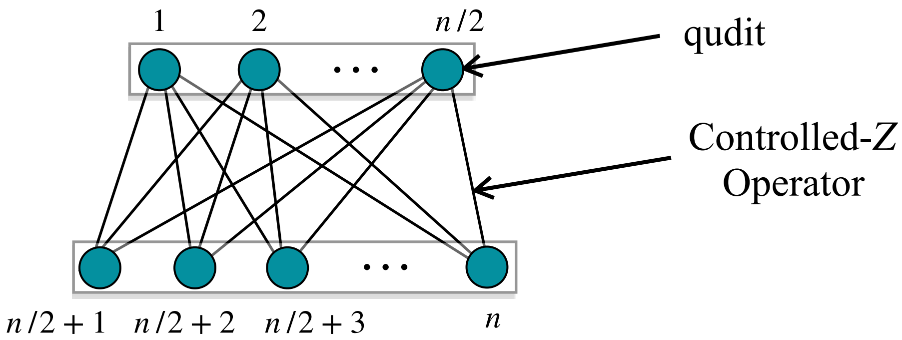

Classical error correcting codes $\longrightarrow$ AME states

$$|\phi\rangle = \sum |\text{all the codewords}\rangle = \sum_i |\vec{c}_i\rangle$$

$$\Gamma = \left[\begin{array}{c|c} 0 & -A \\ \hline -A^T & 0 \end{array}\right]$$

A is a non-singular matrix.

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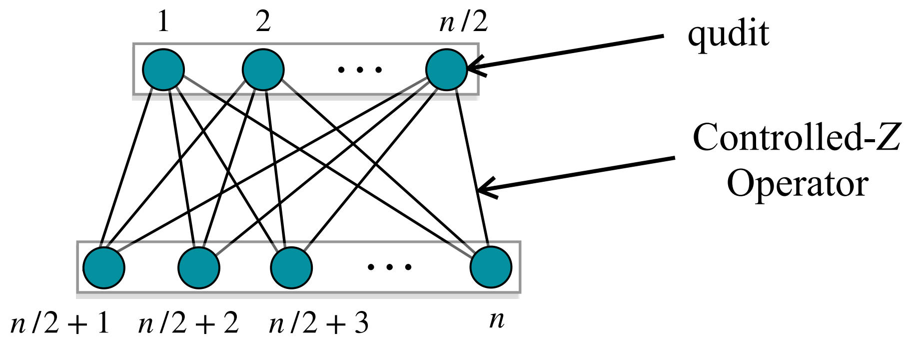

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A is a non-singular matrix.

We substantially broaden the class of known AME states

$$\Gamma = \left[\begin{array}{c|c} 0 & -A \\ \hline -A^T & B \end{array}\right]$$

where B is an arbitrary matrix.

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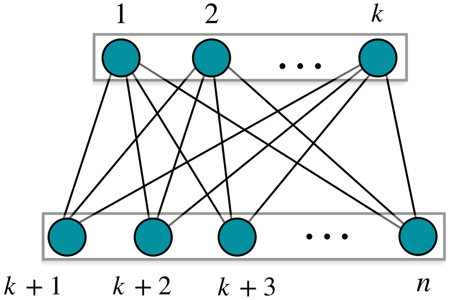
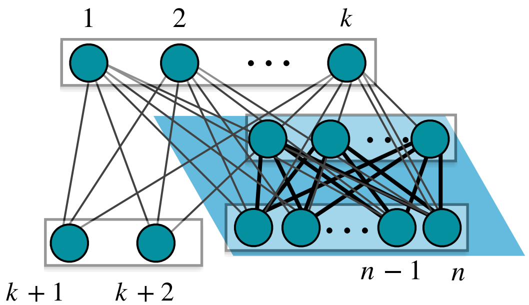
ZR, Adam Burchardt, and Edwin Barnes, Phys. Rev. A **106**, 062424 (2022).

Hierarchical graph states

- A subset is defined by an iterative procedure for constructing a hierarchy of k-uniform graph states.



$$\Gamma^{(n,0)} = \left[\begin{array}{c|c} 0 & -A \\ \hline -A^T & 0 \end{array}\right]$$



$$\Gamma^{(n,n^\star)} = \left[\begin{array}{c|c} 0 & -A \\ \hline -A^T & \begin{array}{c|c} 0 & 0 \\ \hline 0 & \Gamma^{(n^\star,0)} \end{array} \end{array}\right] = \left[\begin{array}{c|c} 0 & -A \\ \hline -A^T & \begin{array}{c|c} 0 & 0 \\ \hline 0 & \begin{array}{c|c} 0 & -A^\star \\ \hline -A^{\star T} & 0 \end{array} \end{array} \end{array}\right]$$

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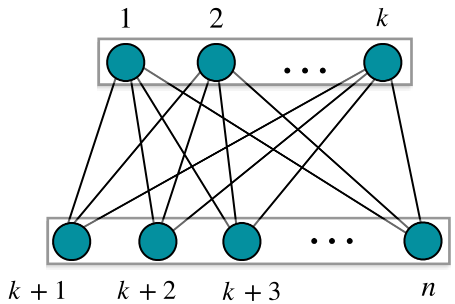

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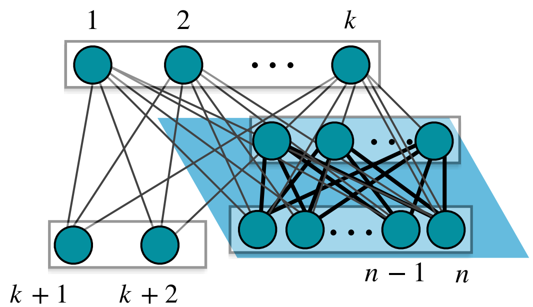

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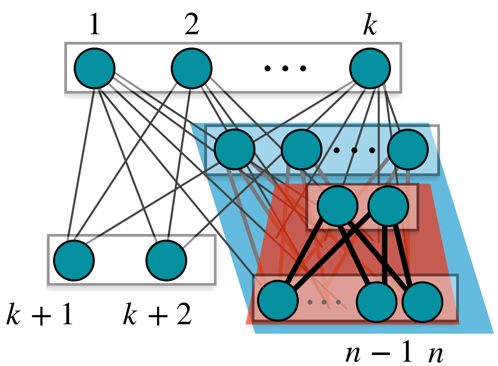

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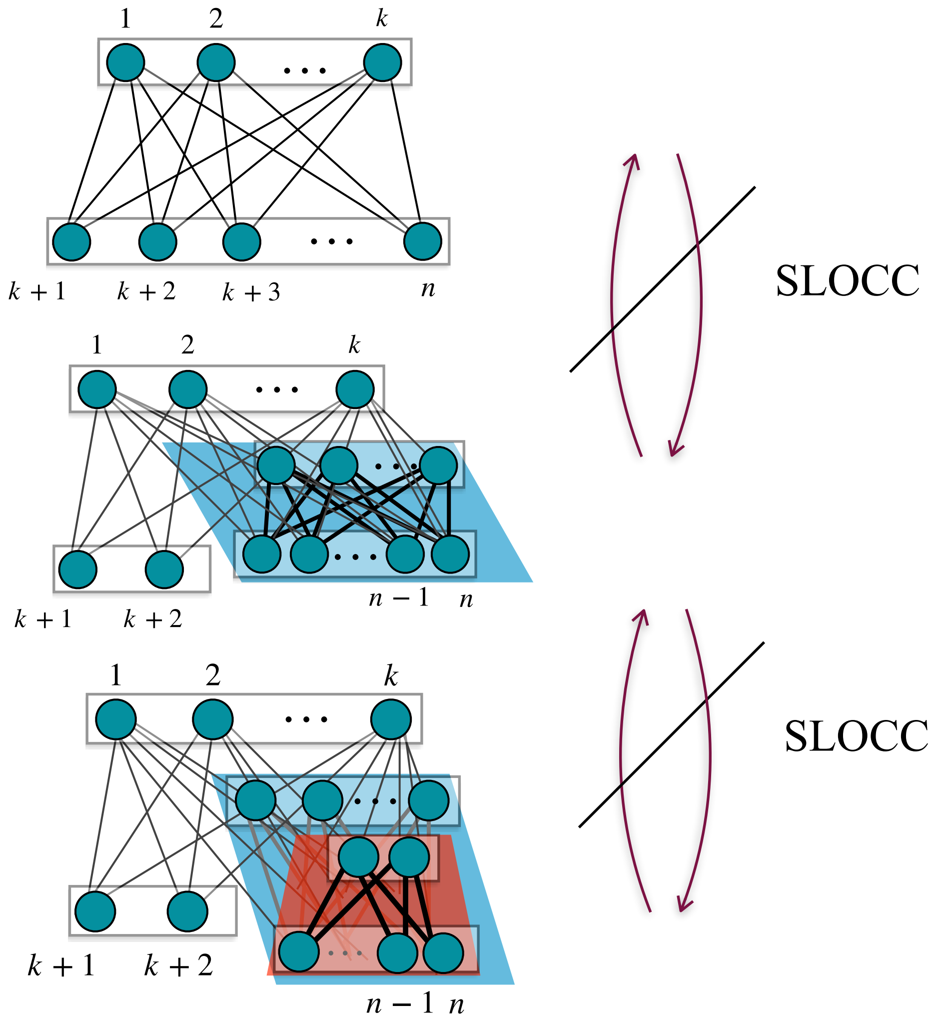
ZR, Adam Burchardt, and Edwin Barnes, Phys. Rev. A **106**, 062424 (2022).

Hierarchical graph states

- AME states (also called perfect tensors) that are not SLOCC equivalent.



- In each level of iteration states are not SLOCC equivalent.

ZR, Adam Burchardt, and Edwin Barnes, Phys. Rev. A **106**, 062424 (2022).

Beyond classical codes

I. Classical part:

$$\vec{c}_i\,, \qquad i \in \{1,\ldots,q^l\}$$

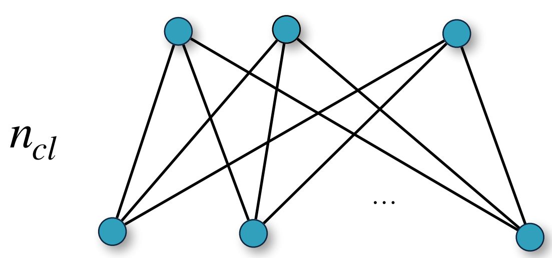
$|\vec{c}_1\rangle$

$|\vec{c}_2\rangle$

$\vdots$

Codewords of classical code

$\underbrace{\hspace{6em}}_{n_{cl}}$



ZR, A. Teixidó, C. Gogolin, A. Acín, Phys. Rev. Research **2**, 033411 (2020), arXiv:1910.12789 [quant-ph]

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II. Quantum part:

$\{\,|\psi_1\rangle, |\psi_2\rangle, \quad \ldots \; |\psi_{q^{n_q}}\rangle\}$

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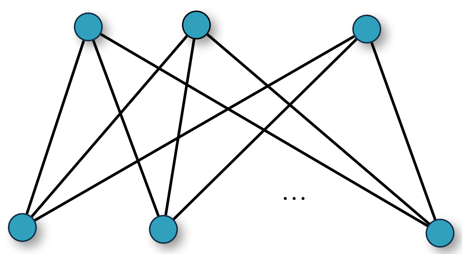
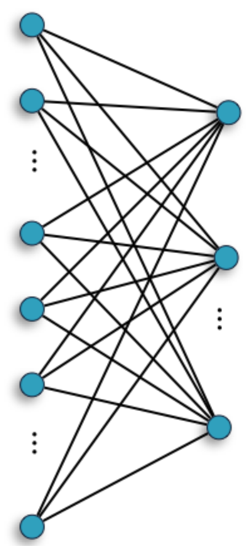
Orthonormal basis

n_q

n_{cl}





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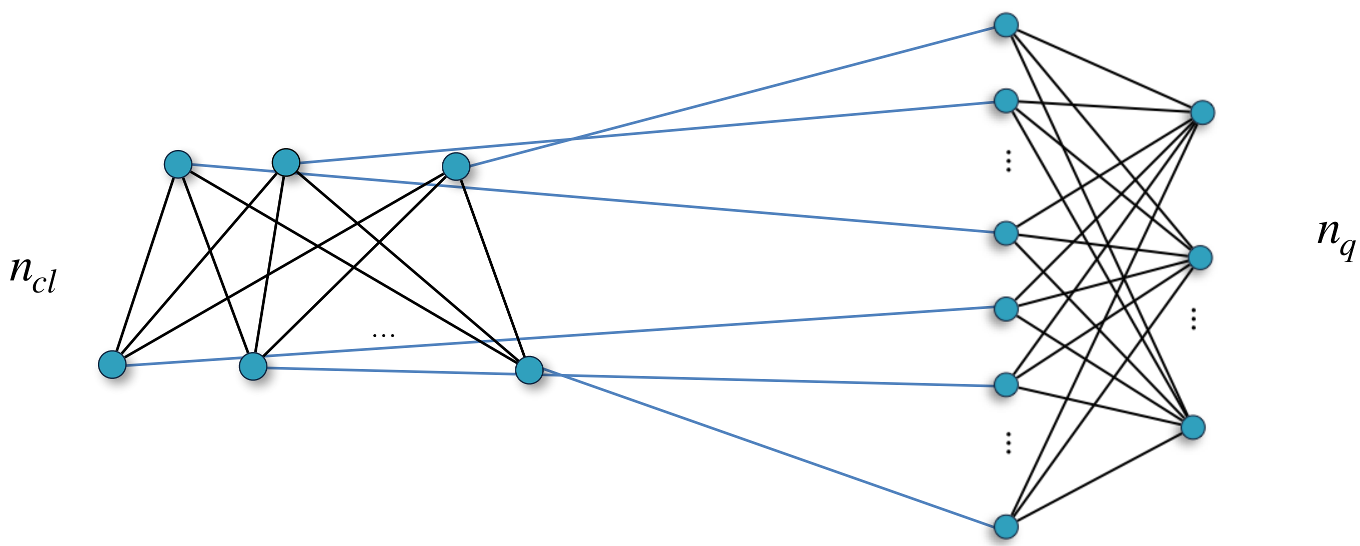

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$\vdots$

Orthonormal basis

$\underbrace{\qquad\qquad\qquad}_{n_q}$

AME state of 5 qubits $\longrightarrow$

$$
\begin{array}{cccccc}
 & & 1 & 2 & 3 & 4\ \&\ 5 \\
|\psi\rangle = & & |0\rangle & |0\rangle & |0\rangle & |\phi^+\rangle \\
 & + & |0\rangle & |1\rangle & |1\rangle & |\psi^+\rangle \\
 & + & |1\rangle & |0\rangle & |1\rangle & |\psi^-\rangle \\
 & + & |1\rangle & |1\rangle & |0\rangle & |\phi^-\rangle
\end{array}
$$

Quantum orthogonal arrays that generalize all previous classes of designs.

where $|\phi^{\pm}\rangle$ and $|\psi^{\pm}\rangle$ are Bell basis

Dardo Goyeneche, **ZR**, Sara Di Martino, Karol Życzkowski, "Entanglement and quantum combinatorial designs", Phys. Rev. A 97, 062326 (2018).

Outline

- Entanglement: resource to overcome LOCC
  - Two-body (bipartite) systems
  - Many-body (multipartite) systems
- Well known many-body entangled classes
- Quantum error correcting codes

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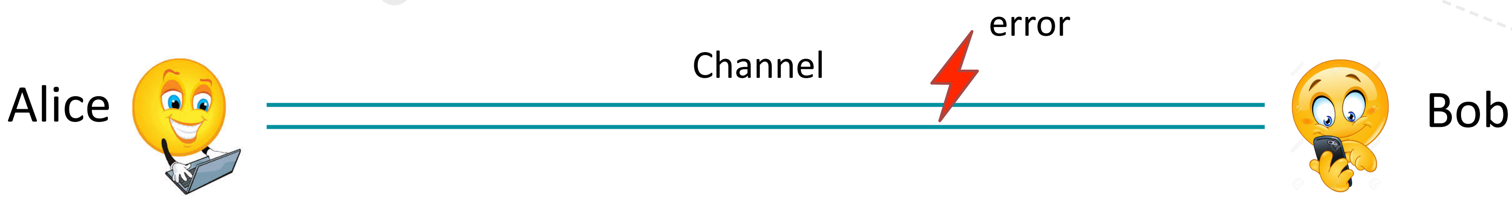
Quantum error correcting codes

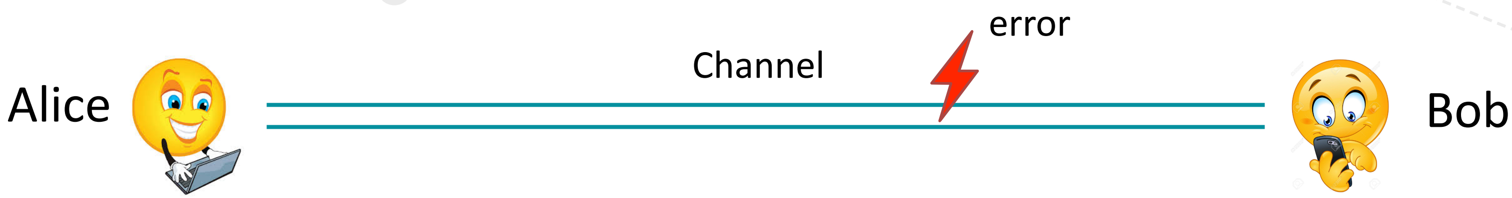


$$[\![n,\ \ k,\ \ d]\!]_q\ ,\qquad d = 2t + 1$$

Message

$|\phi\rangle \in (C_q)^{\otimes k}$

Quantum error correcting codes

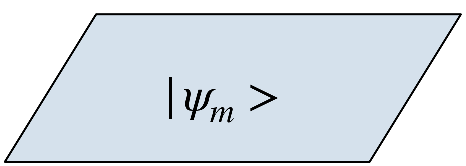


$$[\![n,\ k,\ d]\!]_q\ , \qquad d = 2t + 1$$

Message

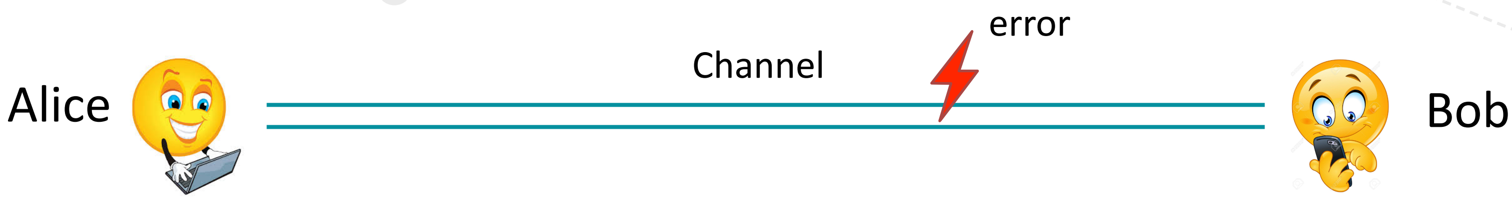
Encoding
(Codewords)

$$|\phi\rangle \in (C_q)^{\otimes k} \longrightarrow |\psi_m\rangle \in \mathscr{C} \subset (C_q)^{\otimes n}$$



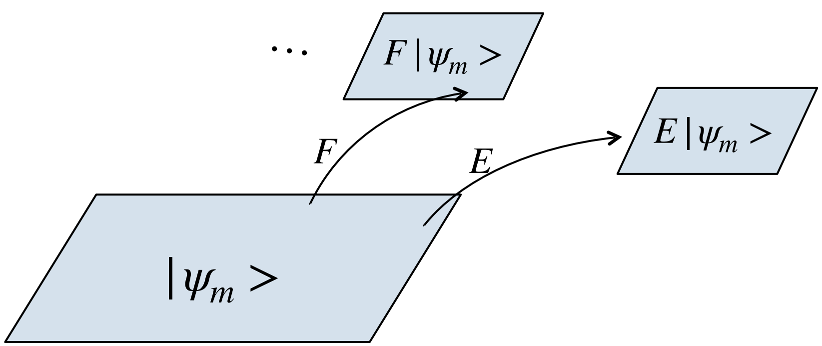
Codespace $\mathscr{C}$

Quantum error correcting codes



$$[\![n,\ k,\ d]\!]_q\ ,\qquad d = 2t + 1$$

Message	Encoding (Codewords)	Error
$\vert\phi\rangle \in (C_q)^{\otimes k}$ $\longrightarrow$	$\vert\psi_m\rangle \in \mathscr{C} \subset (C_q)^{\otimes n}$ $\longrightarrow$	$\vert\tilde{\psi}\rangle \notin \mathscr{C}$



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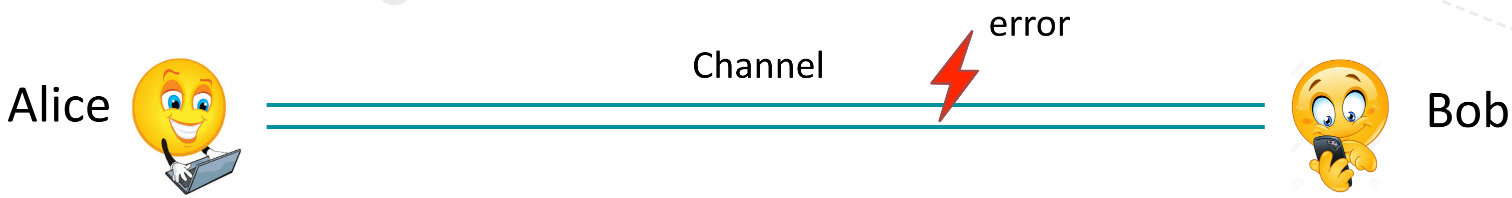
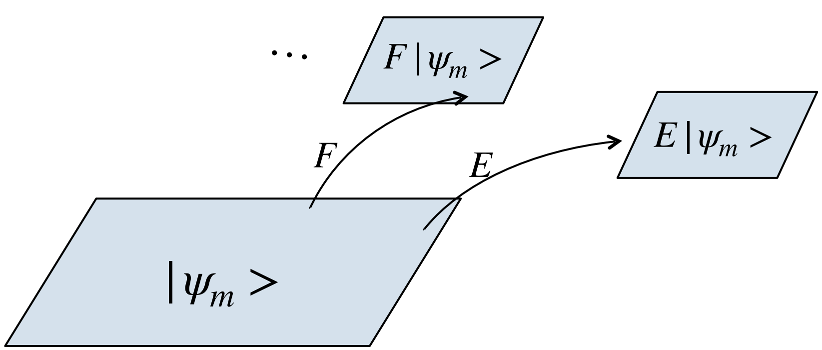


$$[\![n,\ k,\ d]\!]_q\ , \qquad d = 2{\color{red}t} + 1$$

Message	Encoding (Codewords)	Error	Correction
$\vert\phi\rangle \in (C_q)^{\otimes k}$	$\vert\psi_m\rangle \in \mathscr{C} \subset (C_q)^{\otimes n}$	$\vert\tilde{\psi}\rangle \notin \mathscr{C}$	$\vert\psi_m\rangle \in \mathscr{C}$



Codespace $\mathscr{C}$

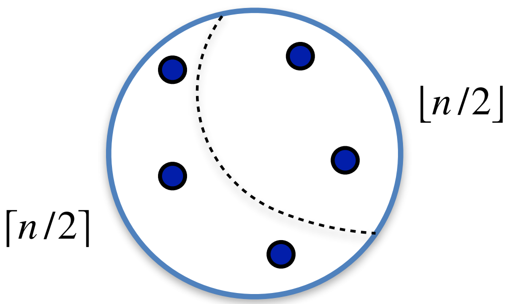
Knill-Laflamme conditions

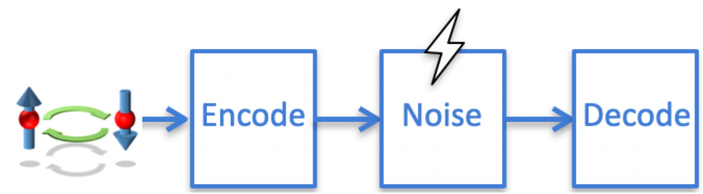
$$<\psi_m|E^\dagger F|\psi_{m'}> = \delta_{m,m'} f(E^\dagger F)$$

- Two different states remain orthogonal
- The function f doesn't depend on $|\psi_m>$

Why k-uniform and AME states?

AME states $\longrightarrow$ Quantum error correcting codes





$$k\text{-uniform states} \equiv [\![n, k=0, k+1]\!]_q$$

$$\text{AME states} \equiv [\![n, k=0, \lfloor n/2 \rfloor + 1]\!]_q$$

Why k-uniform and AME states?

AME states

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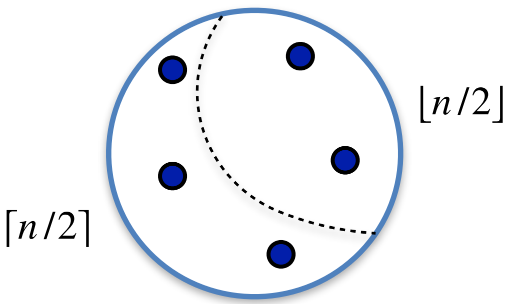

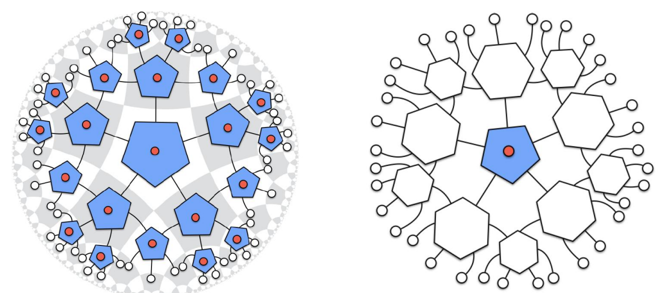

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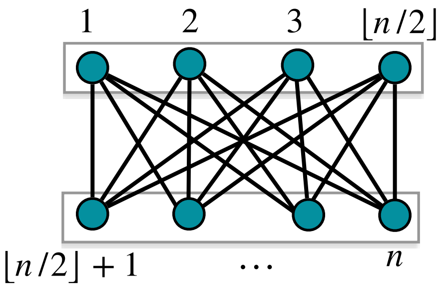
F. Patawski, B. Yoshida, D. Harlow, and J. Preskill, Journal of High Energy Physics 06, 149 (2015).

Shortening process

⇨ Starting k-uniform or AME state and measuring one particle



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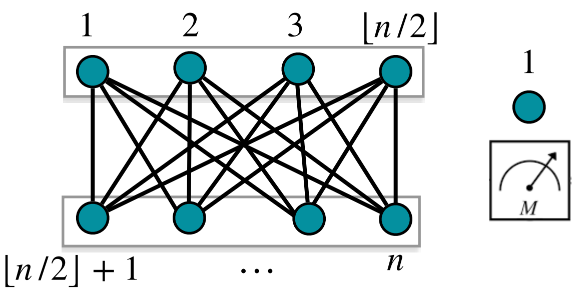

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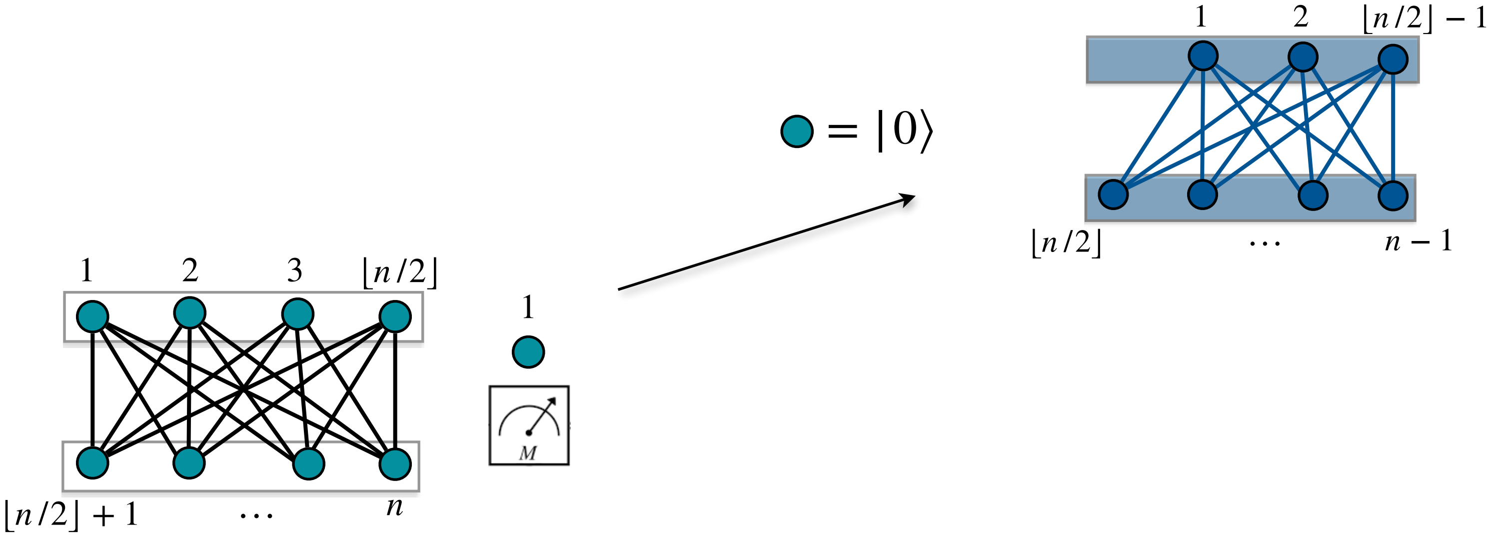

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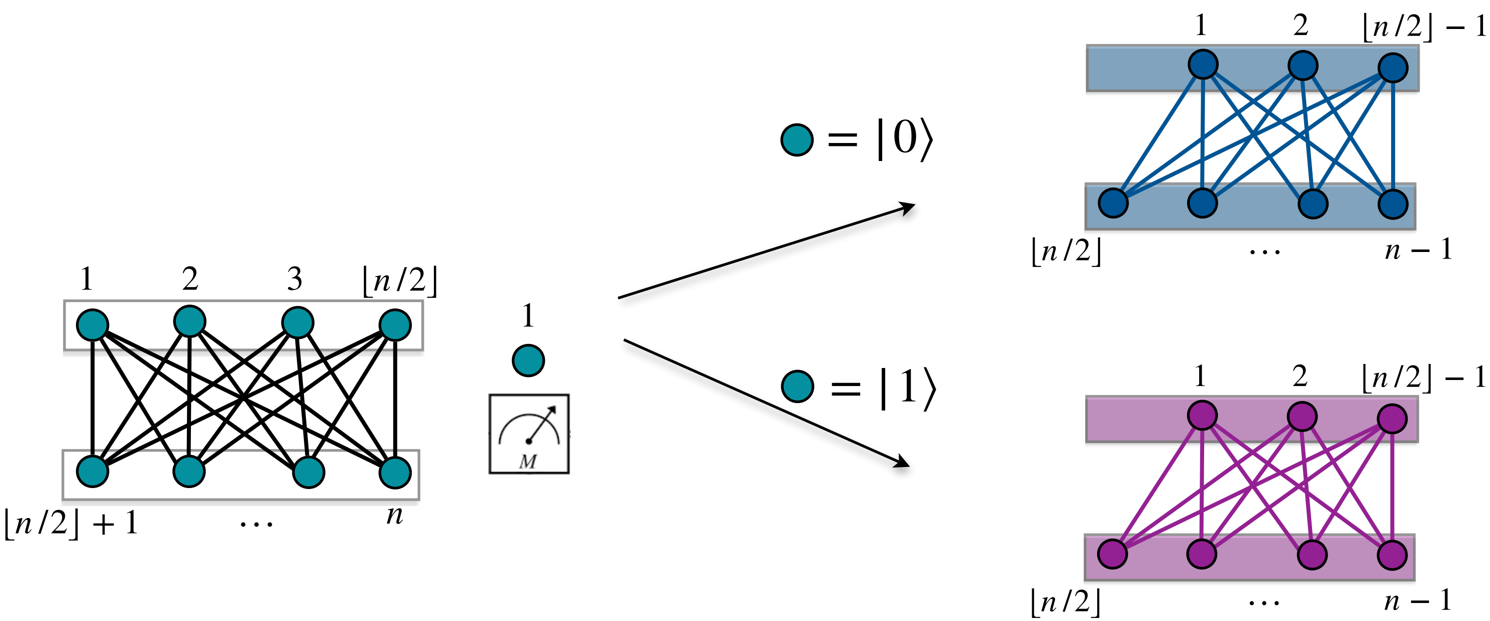

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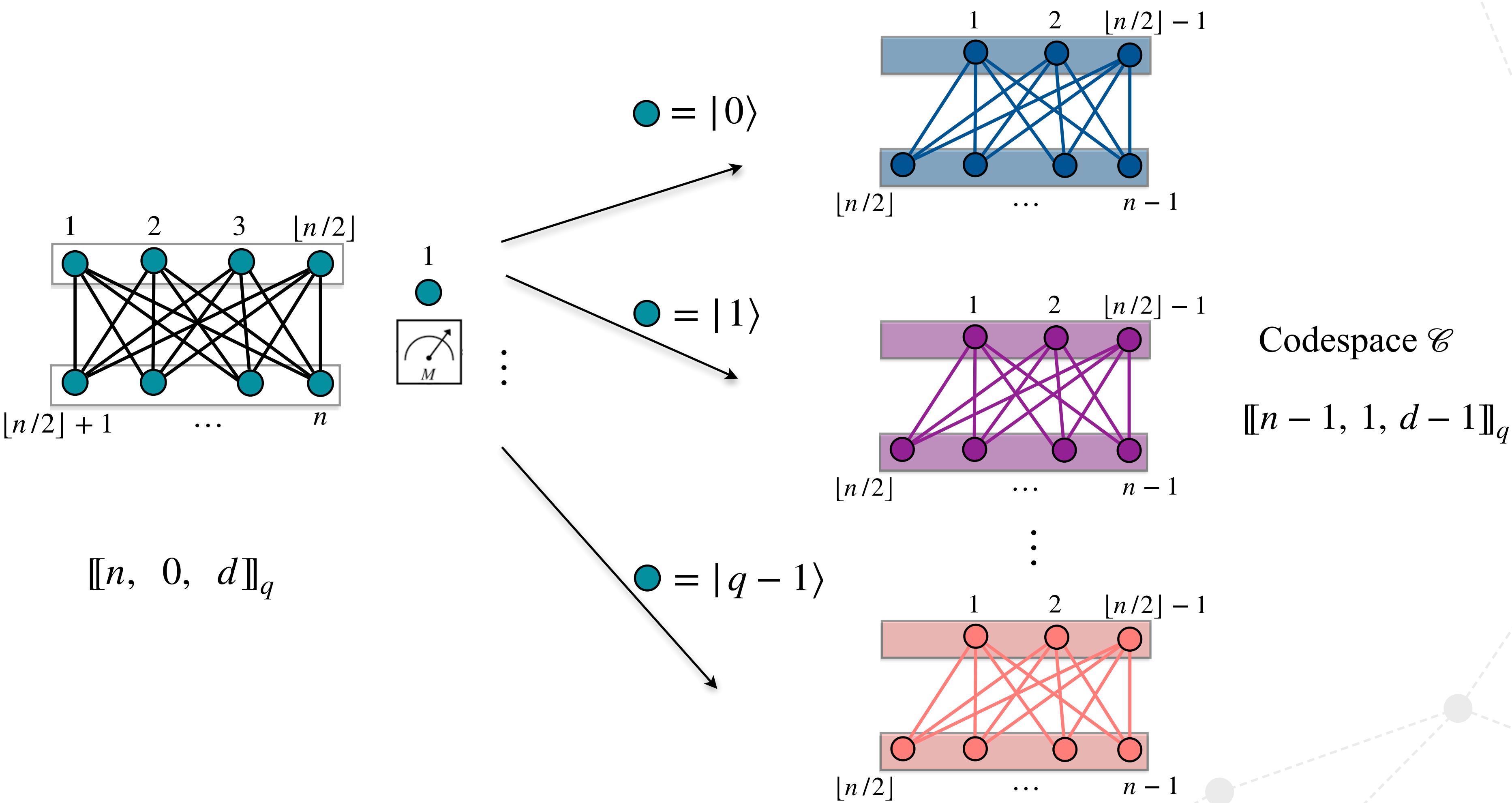
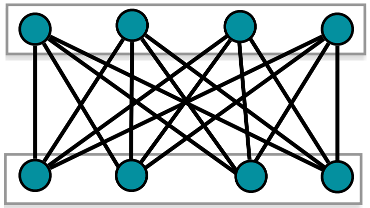


Modified-Shortening process

Starting k-uniform or AME state and performing specific Pauli string

$[\![n,\ \ 0,\ \ d]\!]_q$



- **ZR**, "Modifying method of constructing quantum codes from highly entangled states" IEEE **8** 222439 (2020).

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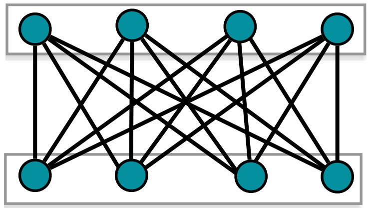

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$M_i = X$ or Z

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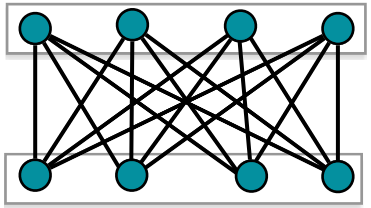

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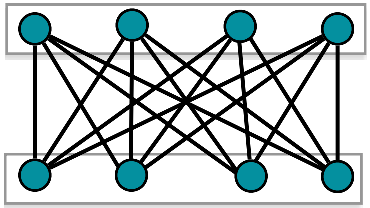

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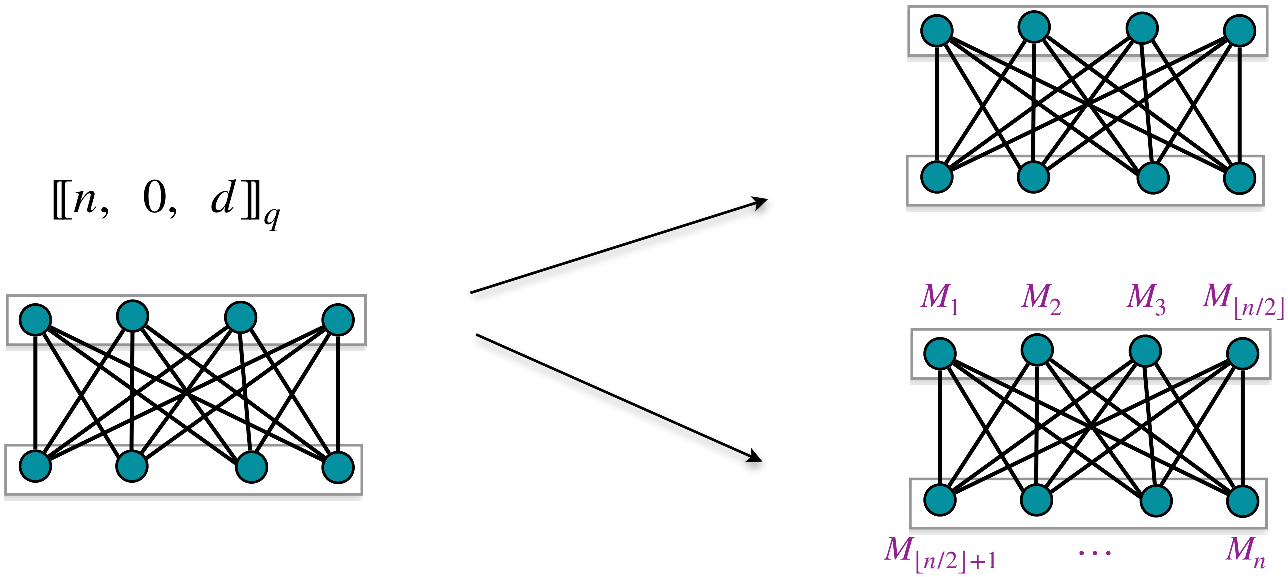

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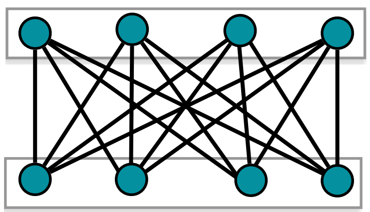

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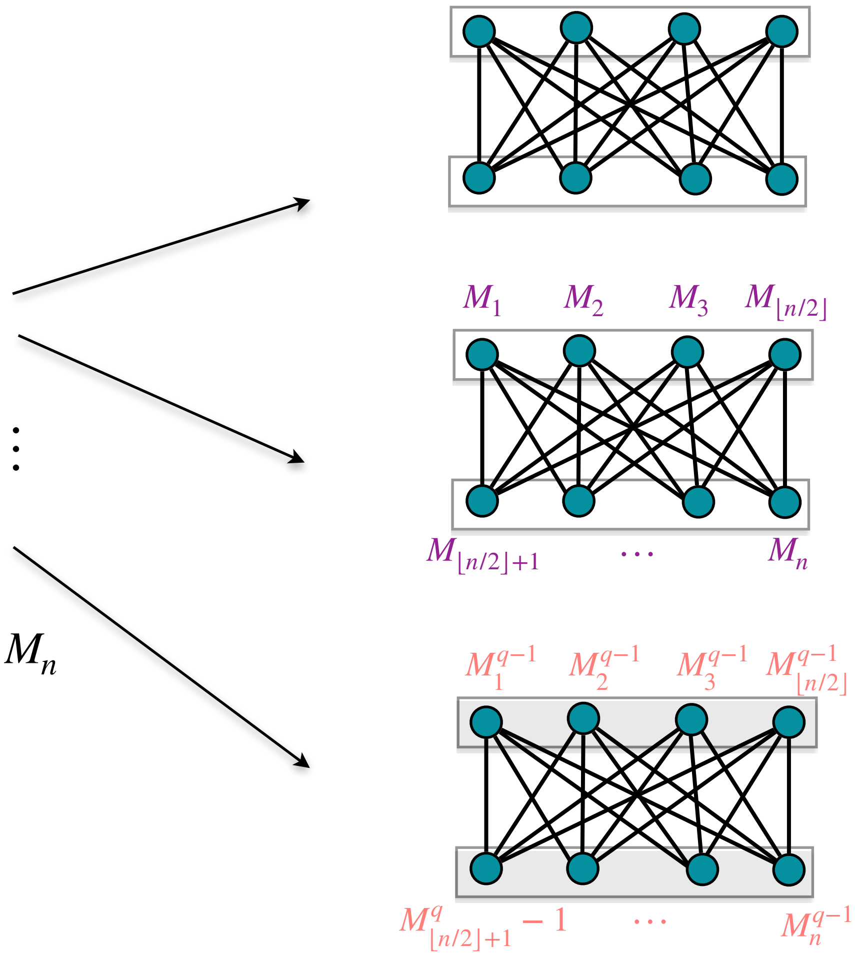

Codespace $\mathscr{C}$

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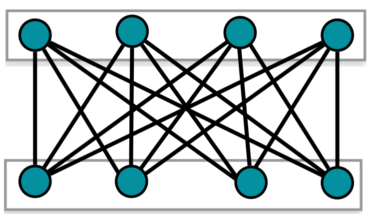

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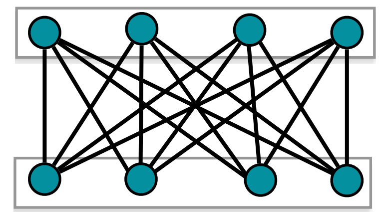

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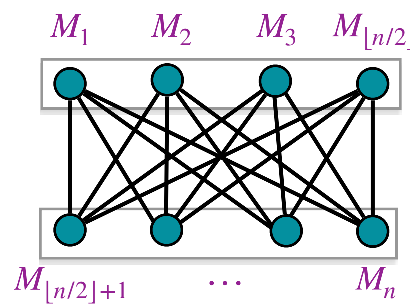
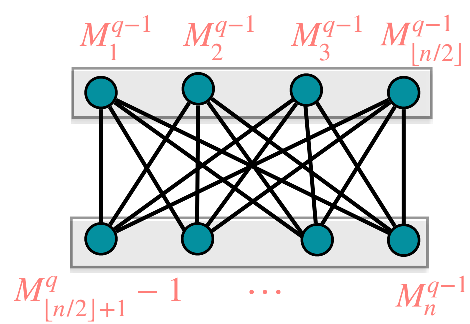


vs. Shortening process



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Comparison

- Comparison between code parameters and subspaces

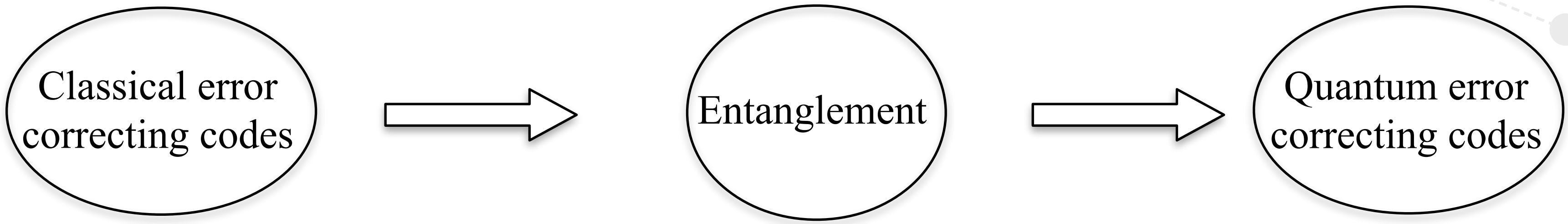
Shortening

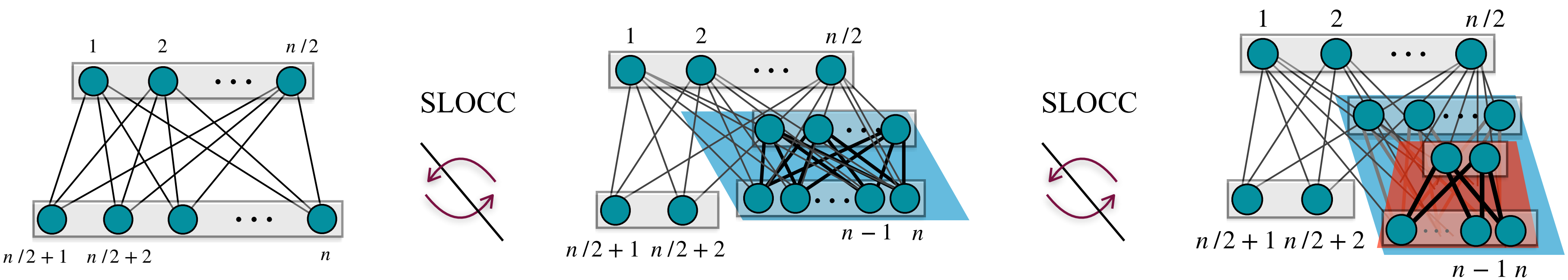
$\mathrm{AME}(n,q)$	$\longrightarrow$	$[\![n-1,1,\lfloor n/2\rfloor]\!]_q$	q-dimensional Subspace
$[\![4,0,3]\!]_{q\geq 3}$	$\longrightarrow$	$[\![3,1,2]\!]_{q\geq 3}$	$\mathrm{AME}(3,q)$
$[\![5,0,3]\!]_{q\geq 4}$	$\longrightarrow$	$[\![4,1,2]\!]_{q\geq 4}$	$1\text{-}\mathrm{UNI}(4,q)$
$[\![6,0,4]\!]_{q\geq 4}$	$\longrightarrow$	$[\![5,1,3]\!]_{q\geq 4}$	$\mathrm{AME}(5,q)$
$[\![7,0,4]\!]_{q\geq 7}$	$\longrightarrow$	$[\![6,1,3]\!]_{q\geq 7}$	$2\text{-}\mathrm{UNI}(6,q)$
$[\![8,0,5]\!]_{q\geq 7}$	$\longrightarrow$	$[\![7,1,4]\!]_{q\geq 7}$	$\mathrm{AME}(7,q)$
$[\![9,0,5]\!]_{q\geq 8}$	$\longrightarrow$	$[\![8,1,4]\!]_{q\geq 8}$	$3\text{-}\mathrm{UNI}(8,q)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$[\![n,0,\lfloor \frac{n}{2}\rfloor+1]\!]_{q\geq n-1}$	$\longrightarrow$	$[\![n-1,1,\lfloor \frac{n}{2}\rfloor]\!]_{q\geq n-1}$	$\lfloor \frac{n-2}{2}\rfloor\text{-}\mathrm{UNI}(n-1,q)$

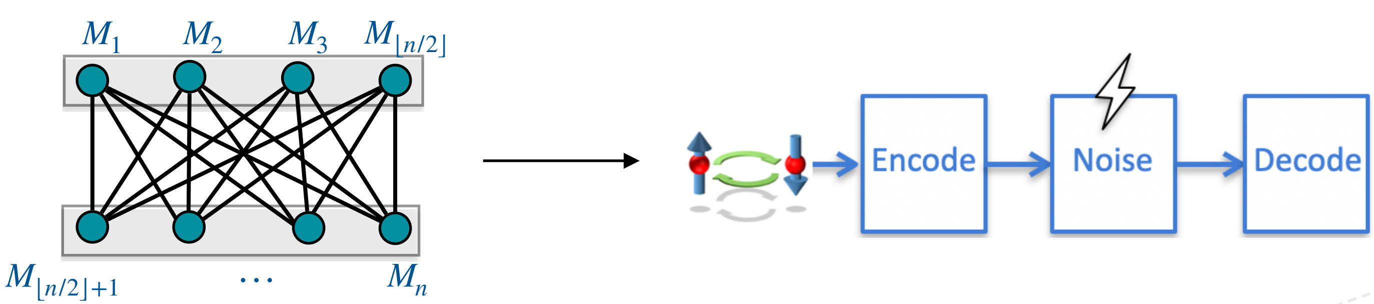
modified-Shortening

$\mathrm{AME}(n,q)$	$\longrightarrow$	$[\![n,1,\lfloor n/2\rfloor]\!]_q$	q-dimensional Subspace
$[\![4,0,3]\!]_{q\geq 3}$	$\longrightarrow$	$[\![4,1,2]\!]_{q\geq 3}$	$\mathrm{AME}(4,q)$
$[\![5,0,3]\!]_{q\geq 4}$	$\longrightarrow$	$[\![5,1,2]\!]_{q\geq 4}$	$\mathrm{AME}(5,q)$
$[\![6,0,4]\!]_{q\geq 4}$	$\longrightarrow$	$[\![6,1,3]\!]_{q\geq 4}$	$\mathrm{AME}(6,q)$
$[\![7,0,4]\!]_{q\geq 7}$	$\longrightarrow$	$[\![7,1,3]\!]_{q\geq 7}$	$\mathrm{AME}(7,q)$
$[\![8,0,5]\!]_{q\geq 7}$	$\longrightarrow$	$[\![8,1,4]\!]_{q\geq 7}$	$\mathrm{AME}(8,q)$
$[\![9,0,5]\!]_{q\geq 8}$	$\longrightarrow$	$[\![9,1,4]\!]_{q\geq 8}$	$\mathrm{AME}(9,q)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$[\![n,0,\lfloor \frac{n}{2}\rfloor+1]\!]_{q\geq n-1}$	$\longrightarrow$	$[\![n,1,\lfloor \frac{n}{2}\rfloor]\!]_{q\geq n-1}$	$\mathrm{AME}(n,q)$

Thanks for your attention

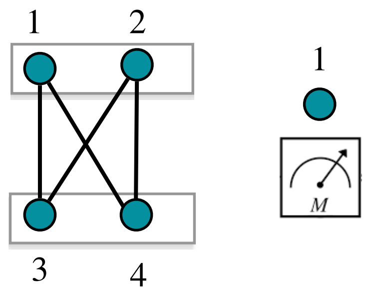




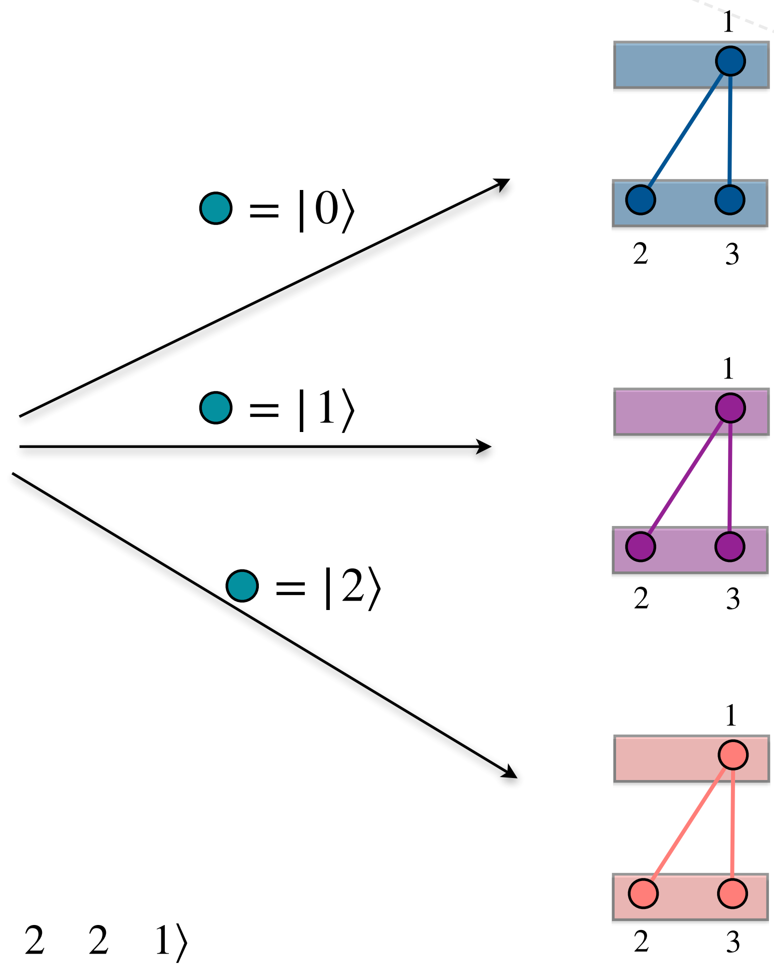


An example

- AME state of 4 qutrit system:



$[\![4,\ 0,\ 3]\!]_{q=3}$

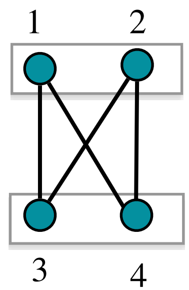


$[\![3,\ 1,\ 2]\!]_{q=3}$

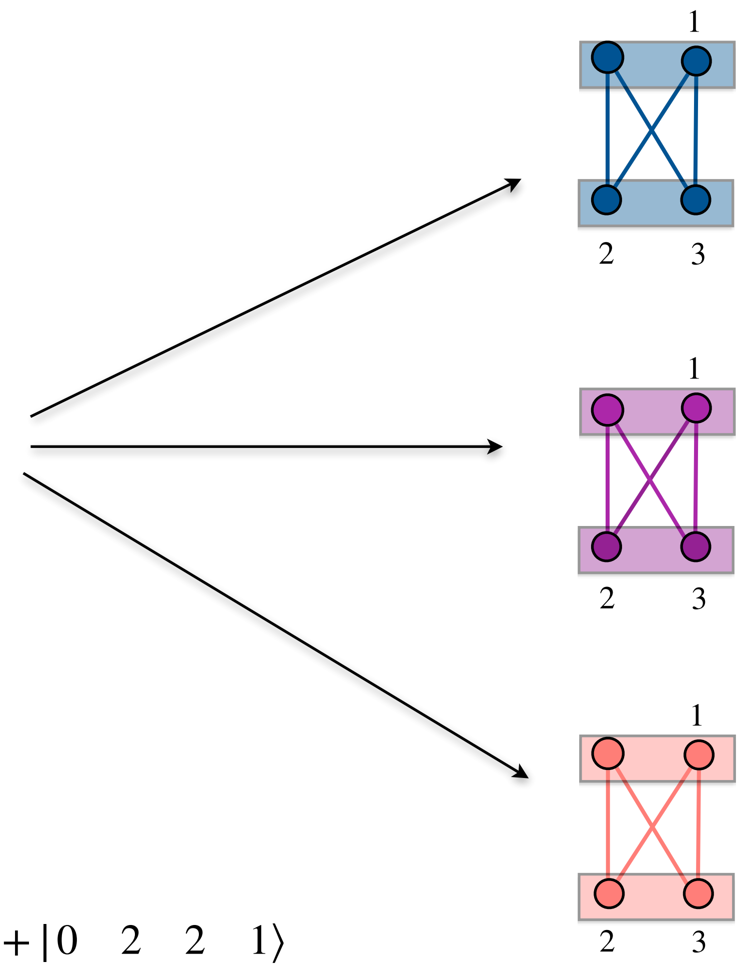
$$
\begin{aligned}
|\psi\rangle &= \sum_{i,j=0}^{2} |i,\ j,\ i+j,\ i+2j\rangle \\
&= |0\ \ 0\ \ 0\ \ 0\rangle + |0\ \ 1\ \ 1\ \ 2\rangle + |0\ \ 2\ \ 2\ \ 1\rangle \\
&\quad + |1\ \ 0\ \ 1\ \ 1\rangle + |1\ \ 1\ \ 2\ \ 0\rangle + |1\ \ 2\ \ 0\ \ 2\rangle \\
&\quad + |2\ \ 0\ \ 2\ \ 2\rangle + |2\ \ 1\ \ 0\ \ 1\rangle + |2\ \ 2\ \ 1\ \ 0\rangle
\end{aligned}
$$

An example

- AME state of 4 qutrit system:



$[\![4,\ \ 0,\ \ 3]\!]_{q=3}$



$M = I \otimes I \otimes X \otimes Z$

$M^2 = I \otimes I \otimes X^2 \otimes Z^2$

$$|\psi\rangle = \sum_{i,j=0}^{2} |i,\ j,\ i+j,\ i+2j\rangle$$

$$= |0\ \ 0\ \ 0\ \ 0\rangle + |0\ \ 1\ \ 1\ \ 2\rangle + |0\ \ 2\ \ 2\ \ 1\rangle$$

$$+\,|1\ \ 0\ \ 1\ \ 1\rangle + |1\ \ 1\ \ 2\ \ 0\rangle + |1\ \ 2\ \ 0\ \ 2\rangle$$

$$+\,|2\ \ 0\ \ 2\ \ 2\rangle + |2\ \ 1\ \ 0\ \ 1\rangle + |2\ \ 2\ \ 1\ \ 0\rangle$$

$[\![3,\ \ 1,\ \ 2]\!]_{q=3}$