Modular Spread/Krylov Complexity

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Outline:

- Motivation
- Krylov basis and quantum complexity measures for operators and states
- Application: Modular Hamiltonian Dynamics
- Conclusions/Open Questions

Based on:

“Quantum chaos and the complexity of spread of states” with V. Balasubramanian, J.M. Magan, Q. Wu, Phys. Rev. D. 106 (2022) 4, 046007

“Geometry of Krylov Complexity” with J.M. Magan, D. Patramanis Phys. Rev. Res. **4**, 013041

Upcoming paper with J.M. Magan (Bariloche) and D. Patramanis (UW)

General Problem

Unitary evolution of states or operators (QM or QFT):

$$i\partial_t |\Psi(t)\rangle = H |\Psi(t)\rangle \qquad \partial_t \mathcal{O}(t) = i[H, \mathcal{O}(t)]$$

$$|\Psi(t)\rangle = e^{-iHt} |\Psi(0)\rangle \qquad \mathcal{O}(t) = e^{iHt}\mathcal{O}(0)e^{-iHt}$$

Generically, a “simple” reference quantum state $|\Psi(0)\rangle$ “spreads” and becomes “complex” (in Hilbert space)

Generically, a “simple” operator $\mathcal{O}(0)$ “grows” and becomes “complex” (in operator space)

How to quantify this Quantum Complexity?

Motivation/Intuition:

$$\mathcal{O}(t) = e^{iHt}\mathcal{O}(0)e^{-iHt} = \mathcal{O}(0) + it[H, \mathcal{O}(0)] + \frac{(it)^2}{2}[H,[H,\mathcal{O}(0)]] + ...$$

E.g.

$$H = \sum_i (Z_i \cdot Z_{i+1} + B_x X_i + B_z Z_i) \qquad \mathcal{O}(0) = X_1$$

2

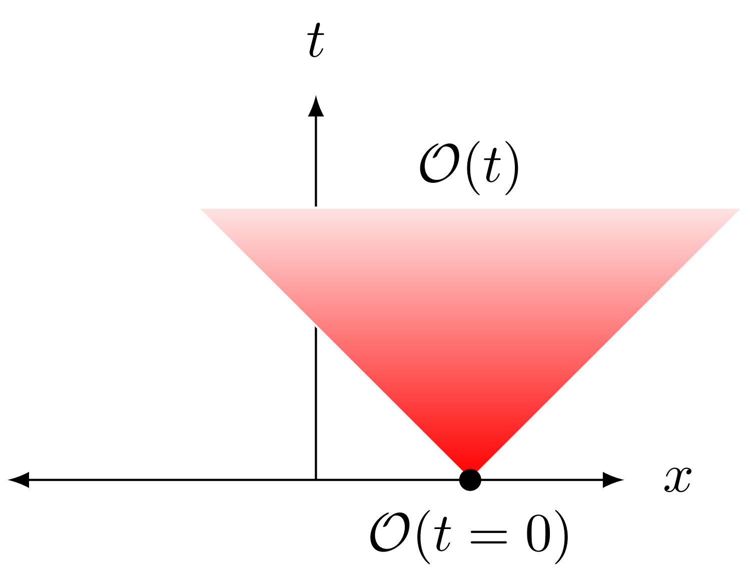
$$\mathcal{O}(t) = X_1 - 2t(Y_1 \cdot Z_2 + B_z Y_1)$$

6

$$-2t^2(B_x Y_1 \cdot Y_2 - B_x B_z Z_1 - B_x Z_1 \cdot Z_2 + 2B_z X_1 \cdot Z_2 + B_z^2 X_1 + X_1 \cdot Z_2^2)$$

13

$$+t^3(..........)$$



Common lore: the more “chaotic” H, the faster the operator grows.

How to quantify this: A universal definition of the operator size/complexity?

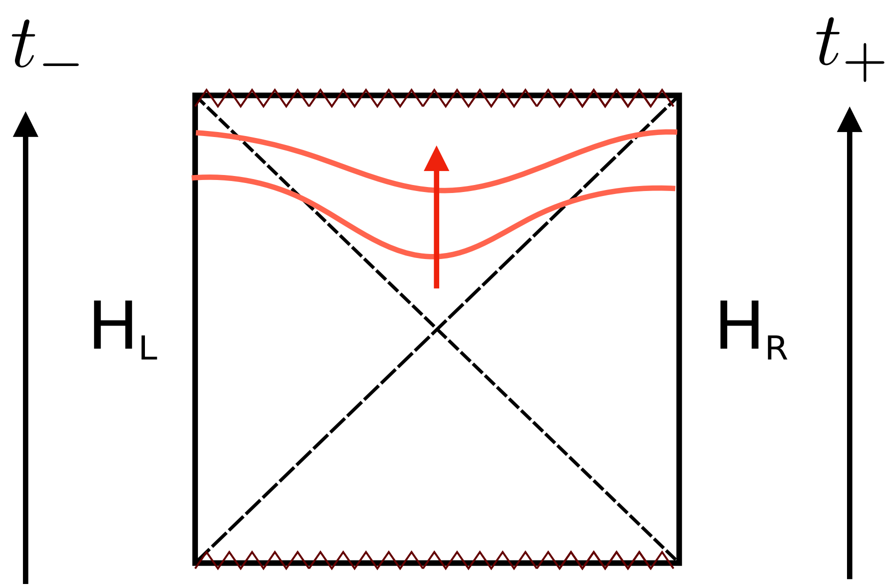
Physics: Definition of Quantum Chaos? ETH, thermalisation…?

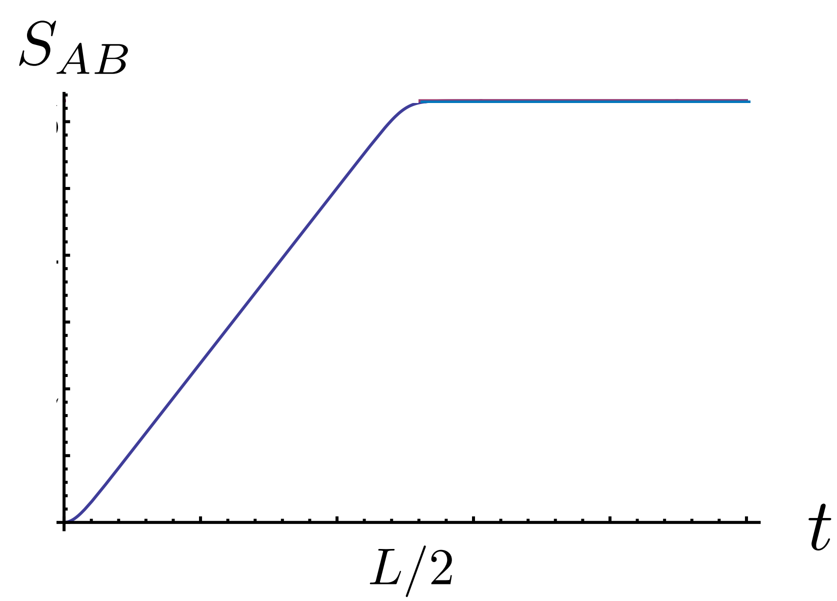
Motivation: Complexity in Holography (HEP)?

[Hartman&Maldacena '13] (2d CFT)

Time-evolved Thermofield-Double state

$$|\Psi_\beta(t)\rangle = e^{-i(H_L+H_R)t}\frac{1}{\sqrt{Z(\beta)}}\sum_n e^{-\frac{\beta}{2}E_n}|n,n\rangle$$





BH (ERB) continues to grow with t but entanglement entropy saturates ("not enough")

What is the "CFT dual" of this (ERB) growth? "Complexity" of the TFD state? [Susskind,'14]

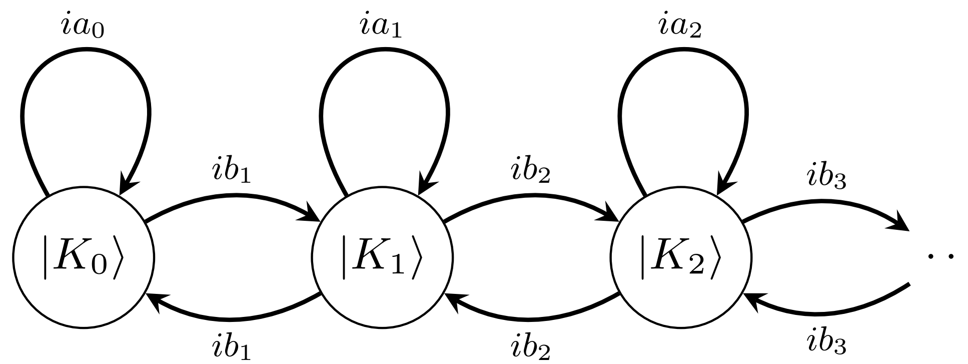
Universal (useful) notion of complexity? Unexplored in QFT (CFT)…

[PC,Kundu,Miyaji,Takayanagi,Watanabe'17] [Jefferson,Myers; Chapman,Heller,Marrochio,Pastawski'17]

[PC,Magan'18] [Flory,Heller'20] [Erdmenger,Flory,Gerbershagen,Heller,Weigel'22]…

Universal framework for quantum complexity?

[Balasubramanian, PC, Magan, Wu ’22]



This talk: describe a notion(s) of quantum complexity based on the Krylov basis

that can be universally defined (and computed) in systems from QM to QFTs

and show some recent results, including Modular Hamiltonian evolution

Basic Idea

Given

$$|\Psi(t)\rangle = e^{-iHt}\,|\Psi(0)\rangle \qquad\qquad \mathcal{O}(t) = e^{iHt}\mathcal{O}(0)e^{-iHt} \equiv e^{i\mathcal{L}t}\mathcal{O}(0)$$

More generally we can think about quantum circuits (circuit H and circuit t)

We can expand them in a certain basis (Krylov basis):

$$|\Psi(t)\rangle = e^{-iHt}\,|\Psi_0\rangle = \sum_n \phi_n(t)\,|K_n\rangle \qquad\qquad |\mathcal{O}(t)) = e^{i\mathcal{L}t}\,|\mathcal{O}_0) = \sum_n \phi_n(t)\,|\mathcal{O}_n)$$

Unitarity: Probability distribution

$$p_n(t) = |\phi_n(t)|^2 \qquad\qquad \sum_n |\phi_n(t)|^2 = 1$$

We will use this probability to characterise the evolution/growth and “complexity”.

Aleksey Nikolaevich Krylov (1863-1945)

Russian naval engineer and applied mathematician.

His mother Sofya Lyapunova came from the famous “Lyapunov" family and Alekandr Lyapunov was his cousin.

He became famous for pioneering “Theory of oscillating motions of the ship”.

In 1904 he built the first machine in Russia for integrating ODEs.

In 1931 he wrote a paper on Krylov subspace: A nxn matrix and b n-vec.

$$\mathcal{K}_r(A, b) = \text{span}\{b, Ab, A^2b, \ldots, A^{r-1}b\}$$

He was interested in efficient diagonalization of matrices and computation of characteristic polynomial coefficients.

“… he was concerned with efficient computations and counted computational work/complexity as the number of separate numerical multiplications ”

Krylov Basis

[Recursion Method: Viswanath,Muller '63]

Unitary evolution/Q-circuit

$$|\Psi(t)\rangle = e^{-iHt}|\Psi_0\rangle = \sum_{n=0}^{\infty}\frac{(-it)^n}{n!}|\Psi_n\rangle$$

Goal: Given states

$$|\Psi_n\rangle \equiv \{|\Psi_0\rangle, H|\Psi_0\rangle, ..., H^n|\Psi_0\rangle, ...\}$$

construct an orthonormal basis $|K_n\rangle$ recursively (Lanczos algorithm, G-S):

$$|A_{n+1}\rangle = (H - a_n)|K_n\rangle - b_n|K_{n-1}\rangle, \qquad |K_n\rangle = b_n^{-1}|A_n\rangle$$

with "Lanczos coefficients":

$$a_n = \langle K_n|H|K_n\rangle, \qquad b_n = \langle A_n|A_n\rangle^{1/2}$$

Such that $b_0 = 0$ and $|K_0\rangle = |\Psi_0\rangle$

Krylov Basis

[Recursion Method: Viswanath,Muller '63]

In the Krylov basis, the Hamiltonian becomes tri-diagonal

$$H|K_n\rangle = a_n|K_n\rangle + b_{n+1}|K_{n+1}\rangle + b_n|K_{n-1}\rangle \qquad \langle K_m|\, H\, |K_n\rangle = \begin{pmatrix} a_0 & b_1 & 0 & 0 & \cdots \\ b_1 & a_1 & b_2 & 0 & \cdots \\ 0 & b_2 & a_2 & b_3 & \cdots \\ 0 & 0 & b_3 & a_3 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

Expanding our state in the Krylov basis

"Hessenberg form"

$$|\Psi(t)\rangle = e^{-iHt}\,|\Psi_0\rangle = \sum_n \phi_n(t)\,|K_n\rangle \qquad \sum_n |\phi_n(t)|^2 \equiv \sum_n p_n = 1$$

By construction, we have a Schrödinger equation for the coefficients (amplitudes)

$$i\partial_t\,|\Psi(t)\rangle = \sum_n i\partial_t\phi_n(t)\,|K_n\rangle$$

$$i\partial_t\,|\Psi(t)\rangle = H\,|\Psi(t)\rangle = \sum_n \phi_n(t)H\,|K_n\rangle = \sum_n \left[a_n\phi_n(t) + b_n\phi_{n-1}(t) + b_{n+1}\phi_{n+1}(t)\right]|K_n\rangle$$

$$i\partial_t\phi_n(t) = a_n\phi_n(t) + b_n\phi_{n-1}(t) + b_{n+1}\phi_{n+1}(t) \qquad \phi_n(0) = \delta_{n,0}$$

Lanczos coef. from return amplitude

[Recursion Method: Viswanath,Muller '63]

[Balasubramanian, PC, Magan, Wu '22]

Lanczos coeff. are encoded in the "return amplitude" (auto-correlator, Loschmidt amp.)

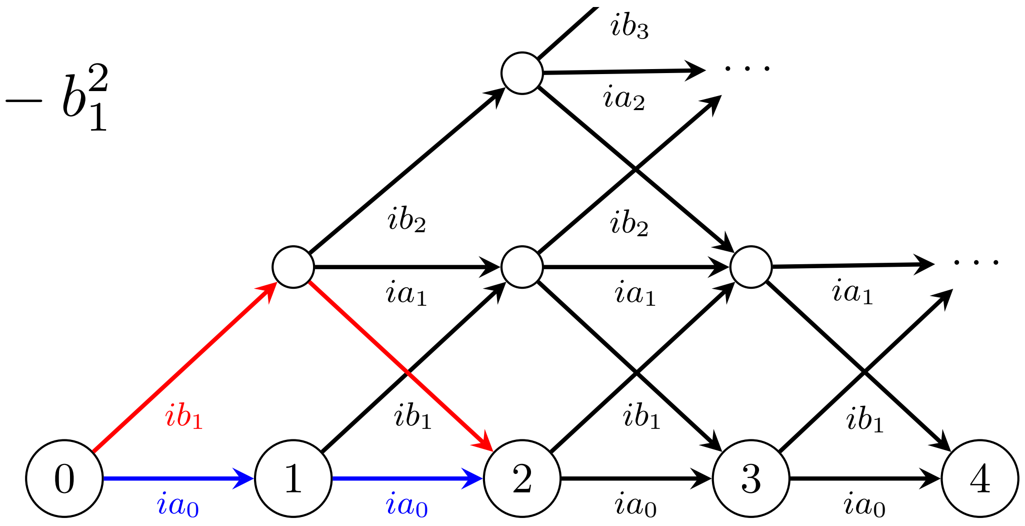
$$S(t) \equiv \langle \Psi(t)|\Psi(0)\rangle = \langle \Psi_0|e^{iHt}|\Psi_0\rangle = \phi_0^*(t)$$

Moments

$$\mu_n = \frac{d^n}{dt^n}S(t)\Big|_{t=0} = \langle \psi(0)|\frac{d^n}{dt^n}e^{iHt}|\psi(0)\rangle\Big|_{t=0} = \langle K_0|(iH)^n|K_0\rangle$$

Knowing moments allows to find Lanczos coefficients (algorithm)

e.g. $\langle K_0|\,(iH)\,|K_0\rangle = ia_0$ $\qquad \langle K_0|\,(iH)^2\,|K_0\rangle = -a_0^2 - b_1^2$



Inverse relations:

$$a_0 = -i\mu_1, \qquad b_1^2 = \mu_1^2 - \mu_2$$

$$a_1 = i\frac{\mu_1^3 - 2\mu_1\mu_2 + \mu_3}{\mu_1^2 - \mu_2} \qquad b_2^2 = \frac{\mu_2^3 + \mu_3^2 + \mu_1^2\mu_4 - 2\mu_1\mu_2\mu_3 - \mu_2\mu_4}{(\mu_1^2 - \mu_2)^2}$$

Physics of Lanczos coeff?
[Balasubramanian, Magan, Wu '22]

Operator Growth in the Krylov Basis

[Recursion Method: Viswanath,Muller ’63]

[Parker, Cao, Avdoshkin, Scaffidi, Altman ’19]

Heisenberg evolution

$$\partial_t \mathcal{O}(t) = i[H, \mathcal{O}(t)] \qquad \mathcal{O}(t) = e^{iHt}\,\mathcal{O}(0)\,e^{-iHt}$$

Formally, we can write the operator as

$$\mathcal{O}(t) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \tilde{\mathcal{O}}_n \qquad \tilde{\mathcal{O}}_0 = \mathcal{O}, \quad \tilde{\mathcal{O}}_1 = [H, \mathcal{O}], \quad \tilde{\mathcal{O}}_2 = [H, [H, \mathcal{O}]], \ldots$$

Liouvillian (super)operator

$$\mathcal{L} = [H, \cdot], \qquad \mathcal{O}(t) \equiv e^{i\mathcal{L}t}\mathcal{O}. \qquad \tilde{\mathcal{O}}_n \equiv \mathcal{L}^n \mathcal{O}.$$

Given $\{\mathcal{O}, \mathcal{L}\mathcal{O}, \mathcal{L}^2\mathcal{O}, \ldots\}$ we need a basis (GNS) $|\mathcal{O})$ $\mathcal{L}\,|\mathcal{O}) = |[H, \mathcal{O}])$

We should pick an inner product:

$$(A|B)^g_\beta = \int_0^\beta g(\lambda)\,\langle e^{\lambda H} A^\dagger e^{-\lambda H} B\rangle_\beta\, d\lambda. \qquad \langle A\rangle_\beta = \frac{1}{Z}\mathrm{Tr}\left(e^{-\beta H} A\right), \qquad Z = \mathrm{Tr}\left(e^{-\beta H}\right)$$

$$g(\lambda) \geq 0, \quad g(\beta - \lambda) = g(\lambda), \quad \frac{1}{\beta}\int_0^\beta d\lambda g(\lambda) = 1.$$

Operator Growth in the Krylov Basis

[Recursion Method: Viswanath,Muller ’63]

[Parker, Cao, Avdoshkin, Scaffidi, Altman ’19]

The most common: Wightman

$$(A|B) = \langle e^{H\beta/2} A^\dagger e^{-H\beta/2} B \rangle_\beta \qquad g(\lambda) = \delta(\lambda - \beta/2)$$

Then we follow the Lanczos algorithm.

Most of the inner products will involve Tr() so we don't need $a_n = 0$

$$|\mathcal{O}(t)) = e^{i\mathcal{L}t}|\mathcal{O}) \equiv \sum_n i^n \varphi_n(t)|\mathcal{O}_n)$$

Schrödinger equation:

$$\boxed{\partial_t \varphi_n(t) = b_n \varphi_{n-1}(t) - b_{n+1}\varphi_{n+1}(t) \qquad \varphi_n(0) = \delta_{n,0}}$$

Lanczos coefficients are encoded in the return amplitude

$$S(t) = (\mathcal{O}_0|\mathcal{O}(t)) = \left(\mathcal{O}_0|e^{i\mathcal{L}t}|\mathcal{O}_0\right) = \varphi_0(t) = \frac{1}{Z}\sum_{n,m} |\langle n|\mathcal{O}|m\rangle|^2 e^{-\left(\frac{\beta}{2}-it\right)E_n} e^{-\left(\frac{\beta}{2}+it\right)E_m}$$

Krylov Basis Summary

States

$$|\Psi(t)\rangle = e^{-iHt}\,|\Psi_0\rangle = \sum_n \phi_n(t)\,|K_n\rangle$$

$$\sum_n |\phi_n(t)|^2 \equiv \sum_n p_n = 1$$

$$i\partial_t\phi_n(t) = a_n\phi_n(t) + b_n\phi_{n-1}(t) + b_{n+1}\phi_{n+1}(t)$$

$$S(t) \equiv \langle\Psi(t)|\Psi(0)\rangle = \langle\Psi_0|e^{iHt}|\Psi_0\rangle = \phi_0^*(t)$$

Operators

$$|\mathcal{O}(t)) = e^{i\mathcal{L}t}|\mathcal{O}) \equiv \sum_n i^n\varphi_n(t)|\mathcal{O}_n)$$

$$\sum_n |\varphi_n(t)|^2 \equiv \sum_n p_n = 1$$

$$\partial_t\varphi_n(t) = b_n\varphi_{n-1}(t) - b_{n+1}\varphi_{n+1}(t)$$

$$S(t) = (\mathcal{O}(0)|\mathcal{O}(t)) = (\mathcal{O}_0|e^{i\mathcal{L}t}|\mathcal{O}_0) = \varphi_0(t)$$

Connections:

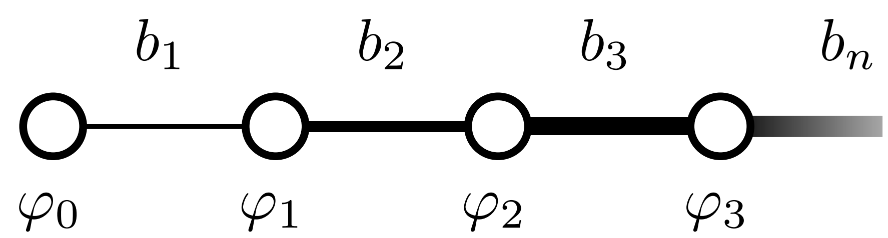
$$|\Psi(t)\rangle = \mathcal{O}(-t)\,|\Psi(0)\rangle$$

E.g. Wightman $|\psi(t)\rangle = \rho_\beta^{1/4}\mathcal{O}_L(t)\rho_\beta^{-1/4}|\psi_\beta\rangle$

Krylov/Spread Complexity

[Parker, Cao, Avdoshkin, Scaffidi, Altman ’19]
[Balasubramanian, PC, Magan, Wu ’22]

The physics of the growth/evolution <=> motion of a particle on a chain



The further in the chain the particle is, the more “complex” state in the Krylov basis needs to be employed (to represent the state or the operator)

A natural definition of “complexity” as an average position on the chain:

$$\mathcal{C}_\Psi(t) = \sum_n n|\phi_n(t)|^2 = \langle\Psi(t)|\hat{K}|\Psi(t)\rangle \qquad \hat{K} = \sum_n n\,|K_n\rangle\langle K_n|$$

Important: Evolution can be characterised with QI/Probability tools:

K-entropy $S_K = -\sum_n p_n \log p_n$ K-variance, K-capacity, $C_K = e^{S_K}$ …

[Barbon, Rabinovici, Shir, Sinha ’19] [PC, Datta ’21] [Patramanis ’21]. ….

Comment: Complexity?

[Balasubramanian, PC, Magan, Wu '22]

Starting from the state: $|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle$

Complexity = "Spread in Hilbert space"

Take a basis: $\mathcal{B} = \{|B_n\rangle : n = 0, 1, 2, \cdots\}$ and a "cost function" (a family, $c_n = n$)

$$C_{\mathcal{B}}(t) = \sum_n c_n |\langle\psi(t)|B_n\rangle|^2 \equiv \sum_n c_n \, p_{\mathcal{B}}(n,t)$$

$$C(t) = \min_{\mathcal{B}} C_{\mathcal{B}}(t)$$

minimum (finite t) for the Krylov basis!

Intuition (Induction): For discrete time evolution, assume N-1 vectors equal to the Krylov basis. Then in the next step:

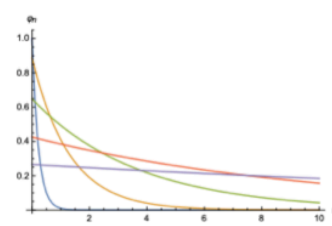
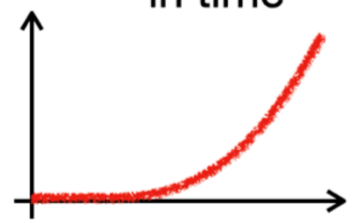
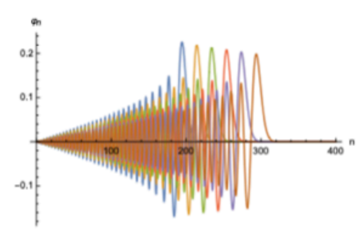
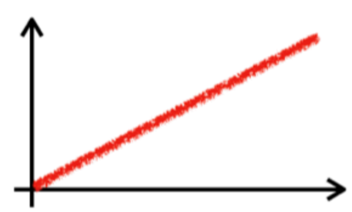
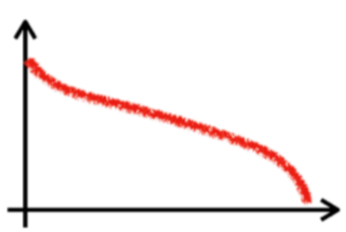
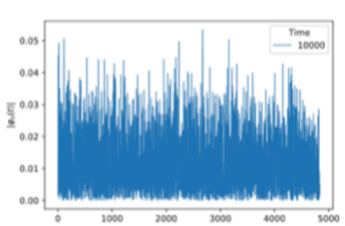
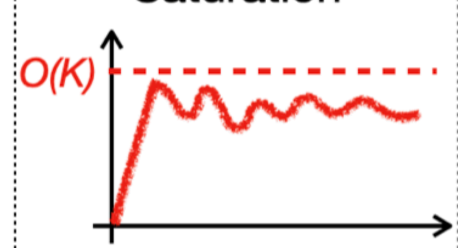
$$|\psi_N\rangle = p_\perp |K_N\rangle + p_{||}|\chi_{||}\rangle$$

Extensive studies of the operator growth

[Parker, Cao, Avdoshkin, Scaffidi, Altman '19]
[Barbon, Rabinovici, Shir, Sinha '19]
[Rabinovici, Sanchez-Garrido, Shir, Sonner '21'22]

Numerics (Operator growth in XXZ chain + Integrability breaking terms, RMT)

S=#dof

n	**Lanczos coefficients**	**wavefunction**	**K-complexity**	**time scales**
$1 \ll n < S$	Linear growth in *n* $b_n \sim \alpha n$		Exponential growth in time 	$0 \lesssim t \lesssim \log S$
$n \gg S$	Plateau, constant in *n* $b_n \sim \Lambda S$		Linear growth in time 	$t \gtrsim \log S$
$n \sim e^{2S}$	Descent 		Saturation 	$t \sim e^{2S}$

Continuum limit: $x = \epsilon n, \quad \varphi(x,t) = \varphi_n(t), \quad v(x) = 2\epsilon b_n = 2\epsilon b(\epsilon n)$

$$\partial_t \varphi(x,t) + v(x)\partial_x \varphi(x,t) + \frac{1}{2} v'(x)\varphi(x,t) = 0$$

(cont. eq for $p = |\varphi|^2$)

Complexity of the TFD evolution

[Balasubramanian, PC, Magan, Wu ’22]

Consider the TFD state

$$|\Psi_\beta\rangle = \frac{1}{\sqrt{Z(\beta)}}\sum_n e^{-\frac{\beta}{2}E_n}|n,n\rangle \qquad Z(\beta)=\sum_n e^{-\beta E_n}$$

and its time evolution

[Hartman,Maldacena ’13]

$$|\psi_\beta(t)\rangle = e^{-iHt}|\psi_\beta\rangle \qquad H = H_L + H_R \qquad H = H_{L/R}$$

Goal: expand this state in the Krylov basis and compute complexity.

Lanczos coefficients from the moments of

$$S(t) = \langle\Psi_\beta(t)|\Psi_\beta\rangle = \frac{Z(\beta - it)}{Z(\beta)}$$

(~SFF)!

[Polchinski et al. ’16]

Non-universal, can be extracted once we know Z (spectrum!).

See [Balasubramanian, PC, Magan, Wu ’22]

Evolution of the TFD for RMT

[Balasubramanian, PC, Magan, Wu '22]
[J. Erdmenger, S-K. Jian, Z-Y Xian '23]

Late Times: "Black Holes and RM" [Polchinski et al. '16]

Consider a random Hamiltonian (NxN, Hermitian matrix, GUE,…)

$$H = \begin{pmatrix} -0.625778 + 0.i & 0.0534572 - 0.238692i & -0.106837 + 0.170713i \\ 0.0534572 + 0.238692i & 0.518485 + 0.i & 0.995288 - 0.0813202i \\ -0.106837 - 0.170713i & 0.995288 + 0.0813202i & -0.589891 + 0.i \end{pmatrix}$$

We can easily diagonalise it, compute SFF, moments, Lanczos, etc.

We want to put it into the tri-diagonal form

and exponentiate

$$\begin{pmatrix} a_0 & b_1 & 0 & 0 & \cdots \\ b_1 & a_1 & b_2 & 0 & \cdots \\ 0 & b_2 & a_2 & b_3 & \cdots \\ 0 & 0 & b_3 & a_3 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

There exist very efficient algorithms/libraries (Python or Mathematica) to put a matrix into this form (Hessenberg). So we can also read off Lanczos coeff. this way.

We also need to "rotate" a TFD into vec: {1,0,0,….}

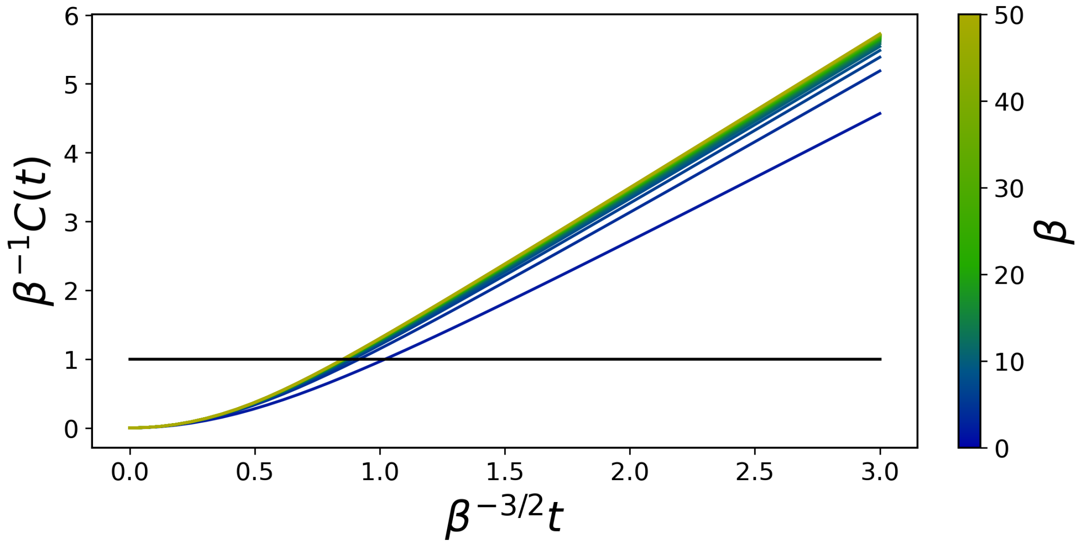
Then applying exp(-iHt) to the initial state gives all the $\phi_n(t)$

Evolution of the TFD for RMT

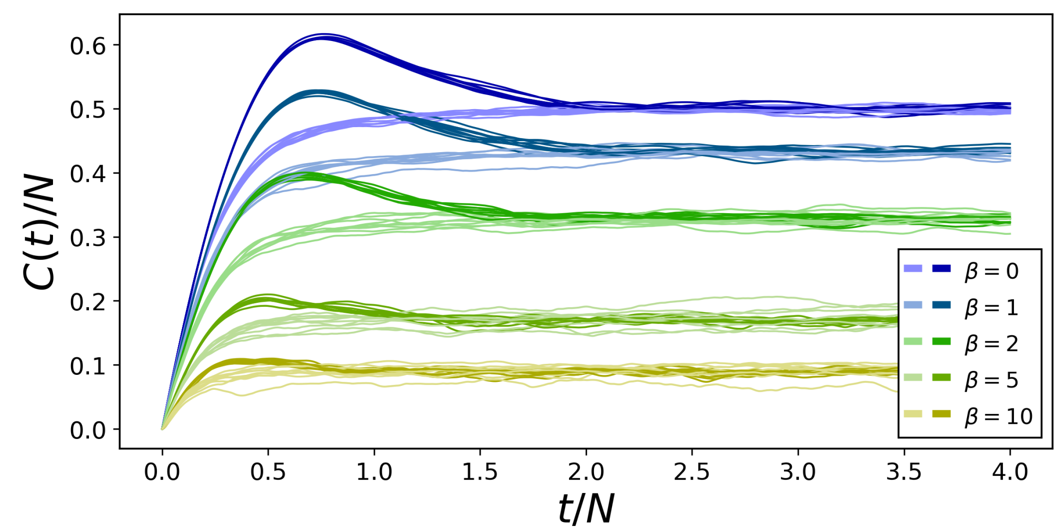
[Balasubramanian, PC, Magan, Wu '22]
[J. Erdmenger, S-K. Jian, Z-Y Xian '23]

Complexity for TFD evolved with GUE Hamiltonian (Similar for GOE,GSE,SYK)

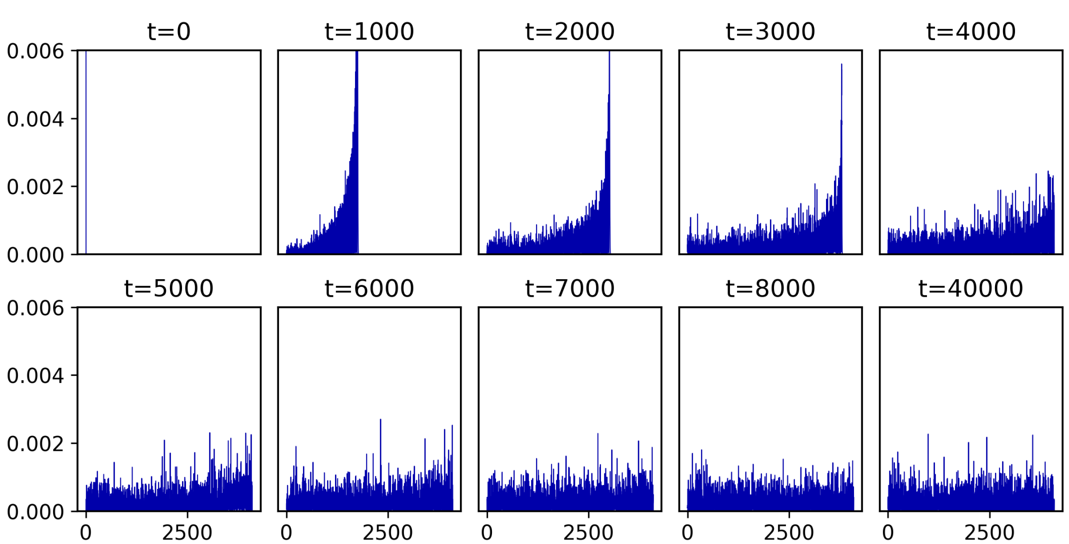
Early time



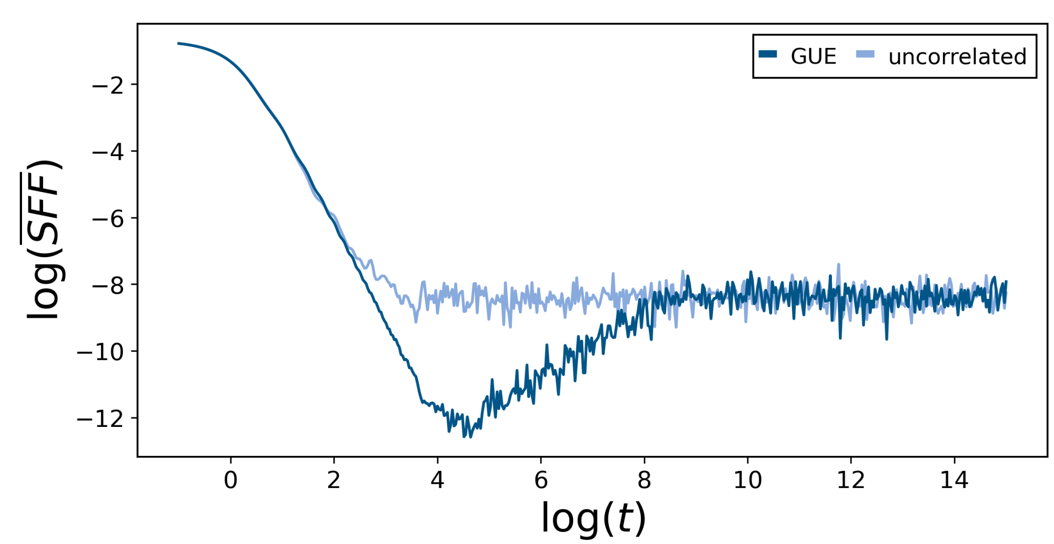
Ramp, Peak, Slope, Plateau



$N = \{1024, 1280, 1536, 1792, 2048, 2560, 3072, 3584, 4096\}$



Slope, Dip, Ramp, Plateau



$N = 4096$ and $\beta = 1$, averaged over 10 samples of the GUE

Motivation: “Modular Hamiltonian”

Setup: $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ $\rho = |\psi\rangle\langle\psi|$

Reduced density matrix: $\rho_A = Tr_B(\rho)$ $\rho_A \equiv e^{-H_A}$ Modular Hamiltonian

“Entanglement spectrum”

$$|\psi\rangle = \sum_n \sqrt{\lambda_n}|n_A\rangle|n_B\rangle \qquad \lambda_n \equiv e^{E_n} \qquad Z(\beta = n) = Tr(\rho_A^n) = \sum_n e^{-nE_n}$$

Much more information than EE. (e.g. topological order…) [Li,Haldane’08]

Modular flow of operators: $\mathcal{O} \in A$

$$\mathcal{O}_s \equiv e^{isH_A}\mathcal{O}e^{-isH_A}$$

Δ^{is} Tomita-Takesaki theory

Operator growth and complexity?

Important AdS/CFT: Bulk reconstruction and bulk locality [Jafferis,Lewkowycz,Maldacena,Suh’15] [Faulkner,Lewkowycz’17]

$$\Phi(X_r) = \int_R dx_R \int ds f^R_{\Delta,s}(X_r|x_R)\mathcal{O}_s(x)\,, \qquad \mathcal{O}_s(x_R) = \rho_R^{-is/2\pi}\mathcal{O}(x_R)\rho_R^{is/2\pi}$$

Spread/Krylov complexity of Modular Evolution?

[PC, J. Magan, D.Patramanis…]

1. States: Modular Spread Complexity

$$|\sqrt{\rho}\rangle = \sum_a \sqrt{\lambda_a}\,|a\rangle_A\,|a\rangle_B \qquad\qquad |\sqrt{\rho}(s)\rangle = e^{-isH_A\otimes 1_B}\left|\rho^{1/2}\right\rangle$$

Return amplitude:

$$S(s) = Tr\left(\rho_A^{1-is}\right) = Z(1-is)$$

or in terms of Renyi entropies

$$S(s) = \exp\left(is\,S_A^{(1-is)}\right) \qquad\qquad S_A^{(n)} = \frac{1}{1-n}\log(\mathrm{Tr}\rho_A^n)$$

Moments and Lanczos coefficients become interesting QI probes:

EE: $a_0 = \langle H_A\rangle = S_A$

Capacity of E: $b_1^2 = \langle H_A^2\rangle - \langle H_A\rangle^2$

Toy example: Qubit

[PC, J. Magan, D.Patramanis…]

$$|\psi\rangle = \sqrt{p}|00\rangle + \sqrt{1-p}|11\rangle$$

Modular Hamiltonian: $\rho_1 = e^{-H_1}$ $\qquad H_1 = \begin{pmatrix} -\log(p) & 0 \\ 0 & -\log(1-p) \end{pmatrix}$

Modular Z: $\mathrm{Tr}(\rho_1^n) = p^n + (1-p)^n$

Return amplitude:

$$S(s) = p^{1-is} + (1-p)^{1-is} = \sum_{k=0}^{\infty} \mu_k \frac{s^k}{k!} \qquad \mu_k = (-i)^k \left(p\log^k(p) + (1-p)\log^k(1-p)\right)$$

Compute Lanczos coeff. and put it in the Krylov basis (tri-diag):

$$\langle K_n|H|K_m\rangle = \begin{pmatrix} -p\log(p) - (1-p)\log(1-p) & \pm\sqrt{p(1-p)}\,(\log(1-p) - \log(p))\,, \\ \pm\sqrt{p(1-p)}\,(\log(1-p) - \log(p))\,, & -p\log(1-p) - (1-p)\log(p) \end{pmatrix}$$

Modular spread complexity:

$$\mathcal{C}(s) = 4p(1-p)\sin^2\left(\frac{s}{2}\log\frac{1-p}{p}\right)$$

Modular flow of operators

[PC, J. Magan, D.Patramanis…]

Total Modular Hamiltonian is well defined in the continuum:

$$\mathcal{O}(s) = e^{isH}\mathcal{O}e^{-isH} \qquad H = H_A \otimes 1_B - 1_A \otimes H_B$$

In 2d CFTs for a single interval A=[a,b] in the vacuum we have (SL(2,R))

$$H = s_{-1}L_{-1} + s_0 L_0 + s_1 L_1 \quad \text{+ bar}$$

Return amplitudes

$$S(s) = \langle \mathcal{O}(s)\mathcal{O} \rangle$$

We can extract modular Krylov complexity

$$C(s) = 2hf(a,b)\sinh^2(\pi s)$$

Universal exponent of the modular growth

Future: Modular chaos from the operator growth?

[de Boer, Lamprou '19]

[de Boer, Jafferis, Lamprou '22]

Conclusions

- New: Krylov/Spread Complexity for operators/states in many-body systems !
- Computable: analytically and numerically for discrete models and QFTs
- New tool for interesting many-body setups (topological phases)
- Crucial ingredient: Return amplitude (2- and higher-point function, SFF etc.)
- Evolution of TFD in RM: Ramp, Peak, Slope, Plateau
- New direction: Spread/Krylov of the modular evolution
- New understanding of entanglement spectra and modular evolution?
- Complexity of local operators in the bulk?

Many Open Problems

- Universal laws for Spread/Krylov complexity? Is it useful for QI or QC?
- Integrable vs Chaotic growth? Is it sensitive? At which time regime?
- Purely Integrable models? Can we study it using integrability (not just numerics)?
- Interesting states? More complicated objects (defects, boundaries)?
- Generalisations: Time dep H(t), Open systems etc.
- Precise connection with Holography? Length in JT [Lin'22, Rabinovici et al. '23]? QGr?
- Late-time physics of AdS/CFT and extremal Black-Holes? [Boruch et al.]

Thank You! Stay Tuned! Join the fun ;)

Spread complexity of formation

[PC, S. Liu '22]

$$|\Psi_\beta\rangle = \frac{1}{\sqrt{Z(\beta)}}\sum_n e^{-\frac{\beta}{2}E_n}|n,n\rangle \qquad H_L = H_R = \omega(\hat{n}+\frac{1}{2}), \qquad E_n = \omega(n+\frac{1}{2})$$

We can write this state as

$$|\Psi_\beta\rangle = e^{ir\tilde{H}}|0,0\rangle \qquad \tilde{H} = \alpha(a_1^\dagger a_1^\dagger + a_1 a_2) \qquad e^{-\beta\omega} = \tanh^2(\alpha r)$$

Action in the eigenstates $|K_n\rangle \equiv |n,n\rangle$

$$\tilde{H}|K_n\rangle = \alpha(n+1)|K_{n+1}\rangle + \alpha n|K_{n-1}\rangle$$

Expansion

$$|\Psi_\beta\rangle = \sum_n i^n \varphi_n(r)|K_n\rangle \qquad \varphi_n(r) = \frac{\tanh^n(\alpha r)}{\cosh(\alpha r)} = \frac{1}{\sqrt{Z(\beta)}}e^{-\frac{\beta}{2}E_n}$$

Krylov complexity (of formation)

$$\mathcal{C} = \sum_n n|\varphi_n(r)|^2 = \sinh^2(\alpha r) = \frac{1}{Z}\sum_n n e^{-\beta E_n} = \frac{1}{e^{\beta\omega}-1} \sim \Delta E$$

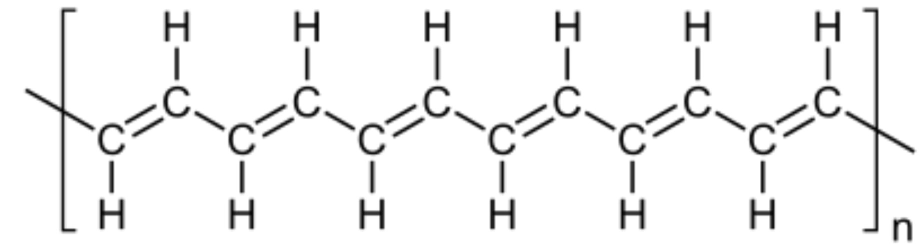
General (e.g. T-matrix and chords in DSSYK, AdS2 length)

[M.Berkooz, P.Narayan, J.Simon'18]
[H.Lin'22][Rabinovici,Sanchez-Garrido,Shir,Sonner'23]

Probe of topological phases?

[PC, S. Liu '22]

SSH model (polyacetylene)



$$H = t_1 \sum_i \left(c^{\dagger}_{Ai} c_{Bi} + \text{h.c.} \right) - t_2 \sum_i \left(c^{\dagger}_{Bi} c_{A,i+1} + \text{h.c.} \right)$$

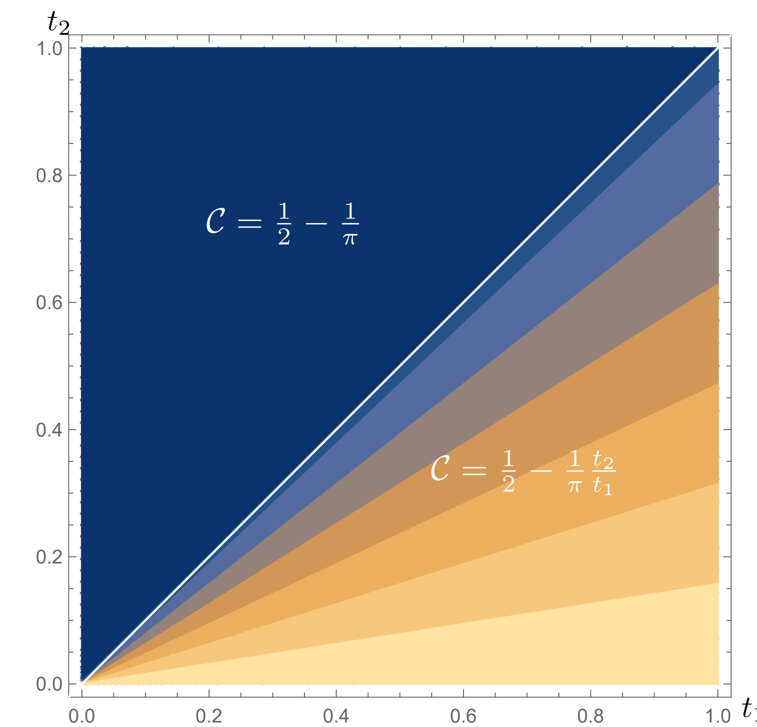
Depending on t's the ground state of the model SU(2) CS:

$$|\Omega\rangle = \prod_{k>0} \mathcal{N}_k e^{-i \tan\left(\frac{\phi_k}{2}\right)(J_+^{(k)} + J_+^{(-k)})} \left| \frac{1}{2}, -\frac{1}{2} \right\rangle_k$$

$$\sin\phi_k = \frac{|R_1|}{R}, \qquad \cos\phi_k = \frac{R_3}{R},$$

$$R_1 = t_1 - t_2 \cos(k) \qquad R_3 = t_2 \sin(k) \qquad R = \sqrt{t_1^2 + t_2^2 - 2 t_1 t_2 \cos(k)}.$$

represents non-topological phase (t1>t2) or topological insulator (t1<t2).



We can use Krylov methods to compute spread complexity of formation for a single momentum and then sum over.

$$\mathcal{C}(t_1, t_2) = 2 \int_0^{\pi} \frac{dk}{2\pi} \mathcal{C}_k = \frac{1}{2} - \frac{t_1 + t_2 - |t_1 - t_2|}{2\pi t_1}.$$

See also Kitaev chain [PC,N. Gupta, S.S.Haque, S. Liu, J. Murugan '22]