Hopf bifurcation in addition-shattering kinetic equations

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Introduction

A well-mixed system undergoing aggregation and fragmentation can be described by equations

$$\frac{dc_k}{dt} = \frac{1}{2}\sum_{i+j=k} K_{ij}c_ic_j - c_k\sum_{j\ge1}K_{kj}c_j + \sum_{j\ge1}F_{kj}c_{j+k} - \frac{1}{2}c_k\sum_{i+j=k}F_{ij}, \quad (1)$$

where $c_k(t)$ is the concentraion of clusters composed of k monomers.
$K_{ij} = K_{ji} \ge 0$ is the rate of aggregation

$$[i]\oplus[j] \xrightarrow{K_{ij}} [i+j] \quad (2)$$

$F_{ij} = F_{ji} \ge 0$ is the rate of binary fragmentation

$$[i+j] \xrightarrow{F_{ij}} [i]+[j] \quad (3)$$

For such class of systems steady oscillations were discovered in [2] and in [3] for exchange-driven aggregation equations.

In the current work (see [1] in detail) we consider a slightly simpler class of processes, and provide much stronger evidence for never-ending oscillations. We consider systems in which each aggregation event involves at least one monomer:

$$[1]\oplus[s] \xrightarrow{A_s} [1+s] \quad (4)$$

and the shattering is assumed to be spontaneous

$$[s] \xrightarrow{B_s} \underbrace{[1]+\cdots+[1]}_{s} \quad (5)$$

The governing equations read

$$\frac{dc_s}{dt} = c_1[A_{s-1}c_{s-1} - A_sc_s] - B_sc_s, \quad s\ge2 \quad (6a)$$

$$\frac{dc_1}{dt} = \sum_{s\ge2}^{\infty} sB_sc_s - 2A_1c_1^2 - c_1\sum_{s\ge2}A_sc_s \quad (6b)$$

and total mass is conserved $M = \sum_{s=1}^{\infty} sc_s(t) \equiv \text{const}$.

Main Objectives

- Demonstrate persistent oscillations for kinetic equations with spontaneous fragmentation,
- Show that they arise via Hopf bifurcation,
- Descrtibe the transition region in parameter space.

Persistent oscillations and choice of coefficients

In this work we concentrate on case $A_s = s$ and recast the original equations into

$$\frac{dc_s}{dt} = c_1[(s-1)c_{s-1} - sc_s] - B_sc_s, \quad s\ge2 \quad (7a)$$

$$\frac{dc_1}{dt} = \sum_{s\ge2}^{\infty} sB_sc_s - c_1^2 - Mc_1 \quad (7b)$$

We assume that total mass is $M = 1$, which can be achieved with proper scaling

$$c_s \mapsto Mc_s, \quad B_s \mapsto MB_s, \quad t \mapsto \frac{1}{M}t \quad (8)$$

These equations are still too general, so we further specialize to a class of algebraic break-up rates

$$B_s = Bs^\beta, \quad \beta \ge 0 \quad (9)$$

All in all, we find the oscillatory solutions in the of parameters (B,β) and show that they arise through a Hopf bifurcation.

Steady states, $0 < \beta < 1$

Suppose $\beta > 0$ and that the system reaches a steady state.
The stationary size distribution obeys $c_s = c_{s-1}(s-1)/(s+B_s/c_1)$, from which

$$\frac{c_s}{c_1} = \prod_{j=2}^{s}\frac{j-1}{j+B_j/c_1} \quad (10)$$

Then we deduce the asymptotic behavior

$$\frac{c_s}{c_1} \propto \begin{cases}(c_1/B)^s(s!)^{-(\beta-1)} & \beta>1\\ s^{-1}\exp[-s^\beta B/\beta c_1] & 0<\beta<1\end{cases} \quad (11)$$

A qualitative change happens at $\beta = 1$ where one can obtain more precise results:

$$\frac{c_s}{c_1} = s^{-1}(1+B/c_1)^{1-s}, \quad c_1 = \frac{\sqrt{B^2+4B}-B}{2} \quad (12)$$

Equation (10) asserts that the stationary size distribution is uniquely determined by the density c_1 of monomers. The mass density

$$M = \sum_{s\ge1} sc_s = c_1\sum_{s\ge1} s\prod_{j=2}^{s}\frac{j-1}{j+B_j/c_1} \quad (13)$$

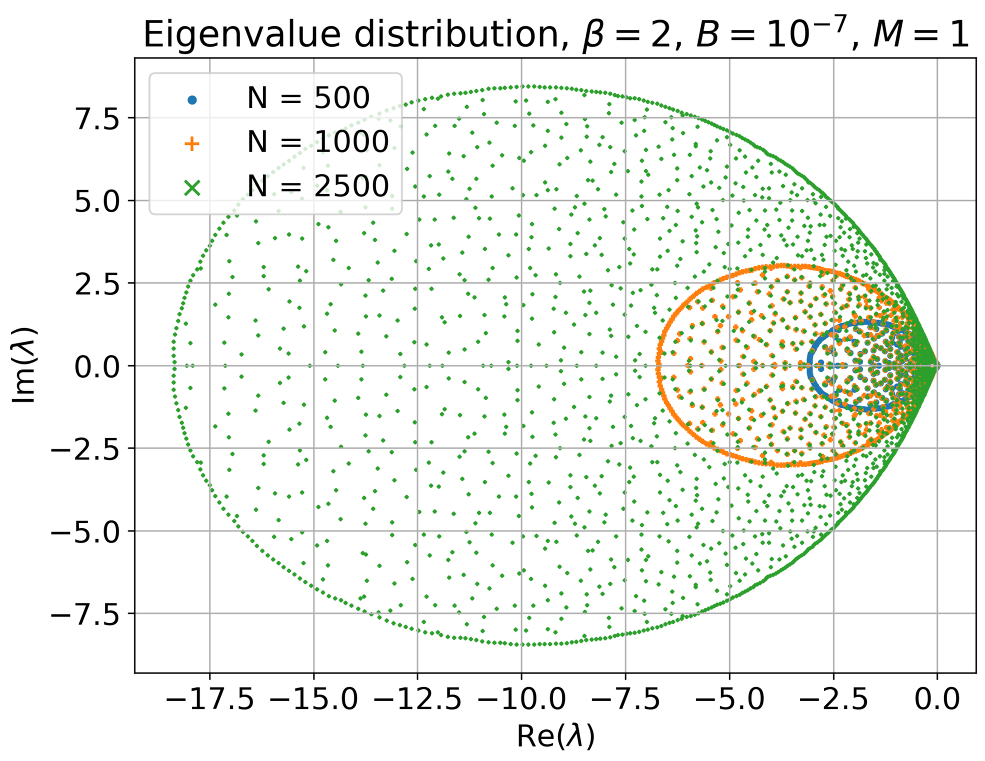
Hopf bifurcation when $\beta > 1$

The stability of the steady state is difficult for theoretical analysis when $\beta > 0$. Owing to mass conservation, the sets of equal-mass size distributions are invariant for (7a)–(7b) and each of them can be considered as phase space (when we talk about the birth of limit cycles we always confine the system to distributions of fixed mass).

To preserve the total mass, we take the following perturbations $(x_s)_{s=1}^{\infty}$ of the steady state:

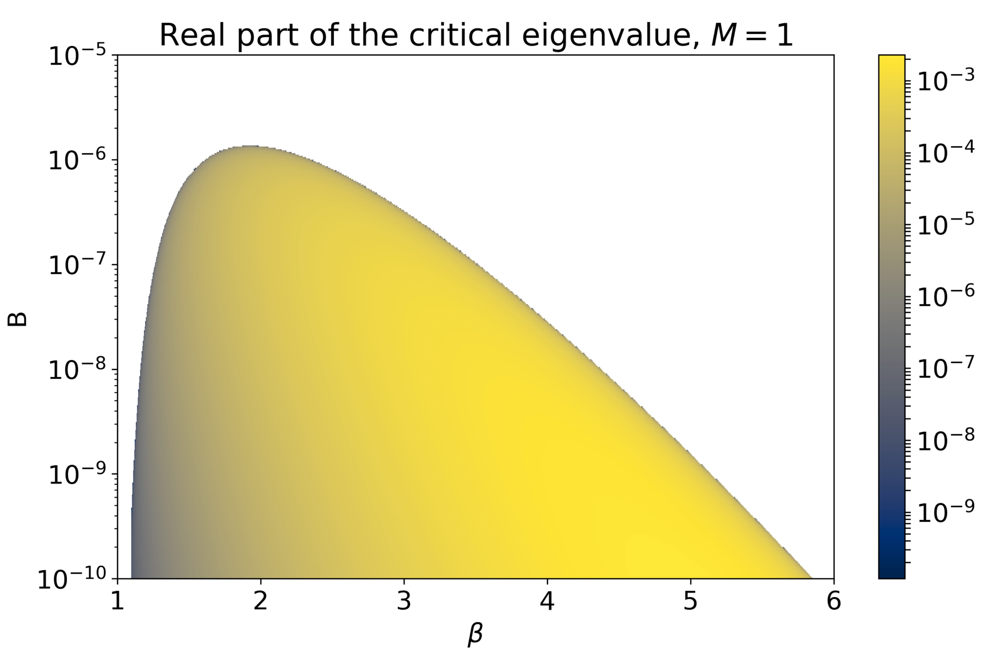
$$\sum_{s\ge1} sx_s(t) = 0. \quad (14)$$

Dropping nonlinear terms in the original equations we arrive at the linearized problem which eigenvalues determine stability of the steady state.

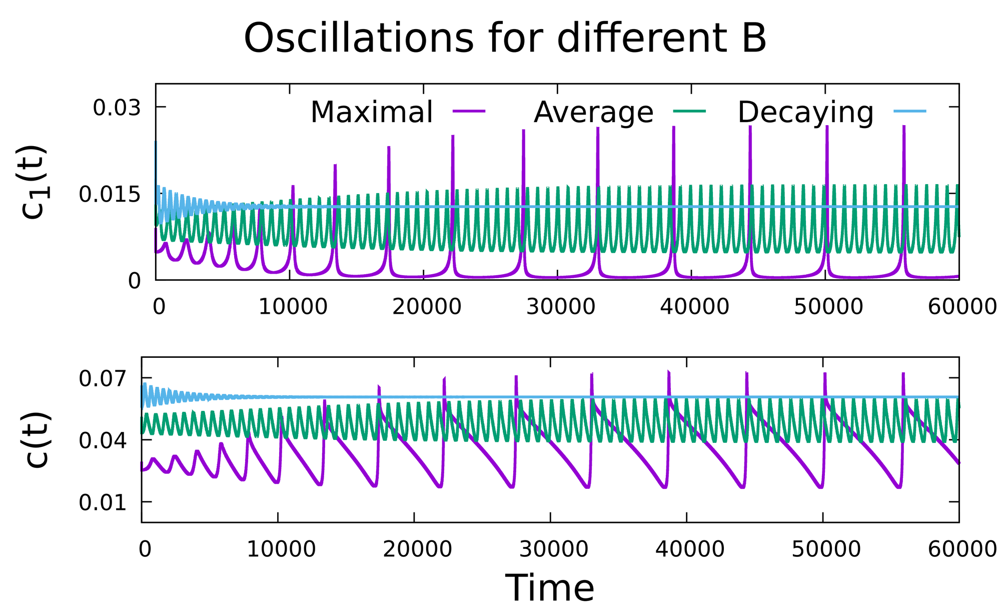


The unstable pairs are present in a certain region in the parameter space

$$\mathcal{U} = \{(\beta,B)\,|\,\beta>1,\ 0<B<B_{\text{crit}}(\beta)\} \quad (15)$$



The steady state loses stability via Hopf bifurcation when B crosses the critical value $B_{\text{crit}}(\beta)$ and enters $\mathcal{U}$. This leads to the birth of a stable limit cycle.



References

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Acknowledgements

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